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Applied to Newsboy Problem**

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ABSTRACT

As a sequel to the original paper by the same authors, Isogai, Ohashi and Sumita (2010), this paper examines the effect of the CDO scheme applied to the classical NBP. The distribution function of the profit with CDO is derived explicitly as a function of the order quantity Q . Sufficient conditions are established under which the optimal solution for the Value at Risk (VaR) problem with CDO is superior or inferior to that without CDO. Furthermore, the VaR problem of NBP without CDO is analyzed in detail for the case of the exponentially distributed demand, deriving the optimal solution Q_{NBP}^{**} and η_{NBP}^{**} explicitly. Assuming that the stochastic demand D is exponentially distributed, extensive numerical experiments reveal that the overall effect of CDO is present when the underlying risk for the opportunity loss is rather large.

Keyword: Collateralized Debt Obligation, Risk Control, Newsboy Problem, Value at Risk

1. Introduction

In the previous paper by the same authors, Isogai, Ohashi and Sumita (2010), the Collateralized Debt Obligation (CDO) scheme has been applied to the classical newsboy problem (NBP) for managing the inventory risk, which arises from uncertainty with respect to the stochastic demand D . The inventory risk was expressed in terms of the opportunity loss between the maximum possible profit and the actual realization of the profit given the order quantity Q . The CDO scheme was then incorporated by specifying a CDO tranche consisting of a pair of an attachment point K_a and a detachment point K_d , as well as the risk-neutral premium ξ . Here, the attachment point K_a means that the protection buyer, which is the retailer issuing the CDO, is fully responsible for the opportunity loss up to K_a . The protection seller, which is the tranche investor buying the CDO, compensates the opportunity loss beyond K_a but up to K_d for the protection buyer. In exchange, the predetermined premium ξ is paid to the protection seller by the protection buyer, where the value of ξ is set in such a way that no-arbitrage condition of the credit derivative market is satisfied, i.e., the expected profit without CDO would be equal to

the expected profit with CDO including the premium payment ξ . Consequently, the expected profit is indifferent about whether or not the CDO scheme is applied.

In order to examine the effect of the CDO scheme applied to NBP, in Isogai, Ohashi and Sumita (2010), the value at risk (VaR) problem for NBP without CDO was first formulated, where the optimal order quantity Q_{NBP}^{**} would be determined so as to minimize the probability η_{NBP}^{**} of the profit falling below v_0 subject to the constraint that the expected profit stays above v_1 . This problem was analyzed mathematically by deriving the distribution function $W_{NBP}(Q, \chi)$ of the profit without CDO, denoted by $PR_{NBP}(Q, D)$, explicitly. In particular, when the stochastic demand is uniformly distributed, both Q_{NBP}^{**} and η_{NBP}^{**} were derived explicitly in a closed form. The VaR problem for NBP with CDO was next formulated with Q_{CDO}^{**} and η_{CDO}^{**} formally defined, and the distribution function $W_{CDO}(Q, \chi)$ of the profit with CDO, denoted by $PR_{CDO}(Q, D)$, was obtained. For the case of the uniformly distributed stochastic demand, numerical results were exhibited, demonstrating under what conditions the CDO scheme would be likely to be effective in that $\eta_{CDO}^{**} < \eta_{NBP}^{**}$, that is, the optimal solution for the VaR problem with CDO is better than that without CDO.

This paper is a sequel to the original paper, where some properties of the CDO scheme applied to NBP are established newly. More specifically, $W_{CDO}(Q, \chi)$ is re-expressed as a function of Q , rather than as a function of χ as in the original paper. This in turn enables one to establish conditions under which a stochastic ordering between $PR_{NBP}(Q, D)$ and $PR_{CDO}(Q, D)$ would be present, that is, stochastically $PR_{CDO}(Q, D)$ is dominated by $PR_{NBP}(Q, D)$ or dominates $PR_{NBP}(Q, D)$. These conditions are also sufficient to assure $\eta_{CDO}^{**} \geq \eta_{NBP}^{**}$ for the former case and $\eta_{CDO}^{**} \leq \eta_{NBP}^{**}$ for the latter case. Furthermore, the VaR problem for NBP without CDO is analyzed for the case of the exponentially distributed stochastic demand. Based on these results, numerical experiments are presented, illustrating the above sufficient conditions and further providing rules of thumb for $\eta_{CDO}^{**} \leq \eta_{NBP}^{**}$ to hold, that is, one can expect the merits of the CDO scheme for the VaR problem with the exponentially distributed stochastic demand.

The structure of this paper is as follows. In Section 2, a succinct summary of the classical NBP and the associated VaR problem with CDO is provided from Isogai, Ohashi and Sumita (2010). The distribution function $W_{CDO}(Q, \chi)$ of $PR_{CDO}(Q, D)$ is then re-expressed as a function of Q in Section 3. Furthermore, sufficient conditions are established under which a stochastic ordering would be present between $PR_{NBP}(Q, D)$ and $PR_{CDO}(Q, D)$. It is shown that these conditions are also sufficient to assure $\eta_{CDO}^{**} \geq \eta_{NBP}^{**}$ (or $\eta_{CDO}^{**} \leq \eta_{NBP}^{**}$). Section 4 is devoted to the detailed analysis of the VaR problem of NBP without CDO for the case of the exponentially distributed demand. Numerical results are given in Section 5, demonstrating that the merits of the CDO approach would exist under certain conditions for the exponential case. Finally, some concluding remarks are given in Section 6.

2. Classical Newsboy Problem and the Associated VaR Problem with CDO

We consider a product whose value drops substantially after a fixed point in time, say τ . The demand for the product over the period $[0, \tau]$ is given as a non-negative random variable D . Throughout the paper, it is assumed that the distribution function of D is absolutely continuous with $F_D(x) \stackrel{\text{def}}{=} P[D \leq x] = \int_0^x f_D(y) dy$ having the mean $\mu_D \stackrel{\text{def}}{=} E[D]$. The corresponding survival function is given by $\bar{F}_D(x) \stackrel{\text{def}}{=} P[D > x] = 1 - F_D(x) = \int_x^\infty f_D(y) dy$.

Let \tilde{c} and \tilde{p} be the procurement cost and the sales price per one product respectively. Given that the order quantity is Q , if $D < Q$, each unsold product has the residual value \tilde{r} . It is natural to assume that

$$0 < \tilde{r} < \tilde{c} < \tilde{p}. \quad (1)$$

If $D > Q$, each of the lost opportunities would cost \tilde{s} . Assuming that the payment would be made and the revenue would be received at time τ , the profit $\tilde{P}R_{NBP}(Q, D)$ can then be described as

$$\tilde{P}R_{NBP}(Q, D) = (\tilde{p} - \tilde{c})Q - (\tilde{p} - \tilde{r})[Q - D]^+ - \tilde{s}[D - Q]^+, \quad (2)$$

where we define $[a]^+ = \max\{a, 0\}$. Let the distribution function and the expectation of $\tilde{P}R_{NBP}(Q, D)$ be denoted by

$$\tilde{W}_{NBP}(Q, x) \stackrel{\text{def}}{=} P[\tilde{P}R_{NBP}(Q, D) \leq x]; \quad \tilde{\pi}_{NBP}(Q) \stackrel{\text{def}}{=} E[\tilde{P}R_{NBP}(Q, D)]. \quad (3)$$

The classical NBP is then to determine the optimal order quantity Q_{NBP}^* so as to maximize $\tilde{\pi}_{NBP}(Q)$. For notational convenience, we write

$$Q_{NBP}^* = \arg \max_Q \tilde{\pi}_{NBP}(Q). \quad (4)$$

From (2), the maximum profit that one can expect is $(\tilde{p} - \tilde{c}) \times D$ which occurs if Q happens to be D . The difference between this maximum profit and the actual profit may then be interpreted as the opportunity loss. More formally, we define

$$\tilde{l}_{NBP}(Q, D) \stackrel{\text{def}}{=} (\tilde{p} - \tilde{c})D - \tilde{P}R_{NBP}(Q, D). \quad (5)$$

Let \tilde{c}_O and \tilde{c}_U be defined by

$$\tilde{c}_O \stackrel{\text{def}}{=} \tilde{c} - \tilde{r}; \quad \tilde{c}_U \stackrel{\text{def}}{=} \tilde{p} - \tilde{c} + \tilde{s}. \quad (6)$$

One sees from (2) and (5) that

$$\tilde{l}_{NBP}(Q, D) = \tilde{c}_O[Q - D]^+ + \tilde{c}_U[D - Q]^+. \quad (7)$$

Let the expectation of $\tilde{l}_{NBP}(Q, D)$ be denoted by

$$\mu_{\tilde{l}:NBP}(Q) = E[\tilde{l}_{NBP}(Q, D)] . \quad (8)$$

It can be readily seen from (3) through (8) that maximizing $\tilde{\pi}_{NBP}(Q)$ is equivalent to minimizing $\mu_{\tilde{l}:NBP}(Q)$. It then follows that

$$Q_{NBP}^* = \arg \min_Q \mu_{\tilde{l}:NBP}(Q) . \quad (9)$$

From (7), it can be shown that

$$\bar{H}_{\tilde{l}:NBP}(Q, x) = P[\tilde{l}_{NBP}(Q, D) > x] = F_D(Q - \frac{x}{\tilde{c}_O}) + \bar{F}_D(Q + \frac{x}{\tilde{c}_U}) . \quad (10)$$

Since $\mu_{\tilde{l}:NBP}(Q) = \int_0^\infty \bar{H}_{\tilde{l}:NBP}(Q, x) dx$, it then follows that

$$\mu_{\tilde{l}:NBP}(Q) = \tilde{c}_O \int_0^Q F_D(x) dx + \tilde{c}_U \int_Q^\infty \bar{F}_D(x) dx . \quad (11)$$

It has been shown that $\mu_{\tilde{l}:NBP}(Q)$ is strictly convex in Q and has the unique minimum point Q_{NBP}^* given by

$$Q_{NBP}^* = F_D^{-1}\left(\frac{\tilde{c}_U}{\tilde{c}_O + \tilde{c}_U}\right) . \quad (12)$$

The reader is referred to Khouja (1999) for further details.

For incorporating the one-term CDO approach in the context of the NBP, it is necessary to convert the monetary values evaluated at time τ , where such values are highlighted by \sim in the above discussions, into the corresponding present values. This can be accomplished by discounting the monetary values evaluated at time τ by $e^{-r_f \tau}$ where r_f is the risk free rate. The present value of a monetary value evaluated at time τ is denoted by dropping \sim in the notation. One can confirm the following conversions.

$$\begin{aligned} p &= e^{-r_f \tau} \tilde{p} ; c = e^{-r_f \tau} \tilde{c} ; r = e^{-r_f \tau} \tilde{r} ; s = e^{-r_f \tau} \tilde{s} ; PR_{NBP}(Q, D) = e^{-r_f \tau} \tilde{P}R_{NBP}(Q, D) ; \\ W_{NBP}(Q, x) &= e^{-r_f \tau} \tilde{W}_{NBP}(Q, x) ; \pi_{NBP}(Q) = e^{-r_f \tau} \tilde{\pi}_{NBP}(Q) ; \\ l_{NBP}(Q, D) &= e^{-r_f \tau} \tilde{l}_{NBP}(Q, D) ; c_O = e^{-r_f \tau} \tilde{c}_O ; c_U = e^{-r_f \tau} \tilde{c}_U . \end{aligned} \quad (13)$$

From (13), it can be readily seen that $\pi_{NBP}(Q)$ achieves the maximum also at Q_{NBP}^* and one has

$$Q_{NBP}^* = \arg \max_Q \pi_{NBP}(Q) = F_D^{-1}\left(\frac{c_U}{c_O + c_U}\right) . \quad (14)$$

It should be noted from (3), (5) and (13) that

$$\pi_{NBP}(Q) = (p - c)\mu_D - \mu_{L:NBP}(Q) , \quad (15)$$

where $\mu_{L:NBP}(Q)$ can be obtained from (11) and (13) as

$$\mu_{L:NBP}(Q) = (c_O + c_U) \int_0^Q F_D(x) dx + c_U(\mu_D - Q) . \quad (16)$$

The next theorem provides a necessary and sufficient condition for the maximum expected profit $\pi_{NBP}(Q_{NBP}^*)$ to be positive.

Theorem 2.1 (Isogai, Ohashi and Sumita (2010))

$$\pi_{NBP}(Q_{NBP}^*) > 0 \text{ if and only if } \frac{s}{c_O + c_U} < \frac{1}{\mu_D} \int_0^{Q_{NBP}^*} s dF_D(x) .$$

Throughout the paper, we assume that the condition of Theorem 2.1 is satisfied.

Recently the classical NBP has been analyzed from the perspective of a conditional VaR problem by Gotoh and Takano (2007). In this paper, as in Isogai, Ohashi and Sumita (2010), the following VaR problem, which is different from that of Gotoh and Takano (2007), is considered.

$$\begin{aligned} & \text{[VaR-NBP]} \\ & \min_Q \eta_{NBP} \text{ subject to } W_{NBP}(Q, v_0) \leq \eta_{NBP} ; \pi_{NBP}(Q) \geq v_1 \end{aligned}$$

From (15), the strict convexity of $\mu_{L:NBP}(Q)$ implies the strict concavity of $\pi_{NBP}(Q)$. Hence, there exist $Q_{v_1:L}$ and $Q_{v_1:R}$ such that the feasible region $FR(v_1)$ for [VaR-NBP] can be written as

$$FR(v_1) = \{Q : \pi_{NBP}(Q) \geq v_1\} = \{Q : Q_{v_1:L} \leq Q \leq Q_{v_1:R}\} . \quad (17)$$

In order to apply the CDO scheme to the classical NBP, we let the loss function $\tilde{l}_{NBP}(Q, D)$ in (7) replace the credit risk in the original CDO context. More specifically, given a tranche consisting of a pair of an attachment point K_a and a detachment point K_d , we define $\tilde{L}_{[K_a, K_d]}(\tau)$ by

$$\tilde{L}_{[K_a, K_d]}(\tau) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \tilde{l}_{NBP}(\tau) \in [0, K_a] \\ \tilde{l}_{NBP}(\tau) - K_a & \text{if } \tilde{l}_{NBP}(\tau) \in [K_a, K_d] \\ K_d - K_a & \text{if } \tilde{l}_{NBP}(\tau) \in [K_d, \infty] \end{cases} , \quad (17)$$

where $\tilde{L}_{[K_a, K_d]}(\tau)$ is the payment paid to the retailer issuing the CDO by the tranche investor buying the CDO at time τ . After a little algebra, as given in Theorem 5.1 of Isogai, Ohashi and Sumita (2010), one finds that $\mu_{L[K_a, K_d]}(Q, D) = E[\tilde{L}_{[K_a, K_d]}(Q, D)]$ is given by

$$\mu_{L[K_a, K_d]}(Q) = \int_{K_a}^{K_d} \left\{ F_D(Q - \frac{x}{c_O}) + \bar{F}_D(Q + \frac{x}{c_U}) \right\} dx . \quad (18)$$

The risk neutral CDO premium ξ is then given by

$$\xi(Q) = \frac{e^{-rf\tau} \mu_{\tilde{L}[K_a, K_d]}(Q)}{K_d - K_a}, \quad (19)$$

where we write $\xi(Q)$ in place of ξ to emphasize that ξ is a function of Q .

Let $PR_{CDO}(Q, D)$ and $PR_{NBP}(Q, D)$ be the present value of the profit with CDO and that without CDO respectively. From (19), the present value of the amount to be paid to the tranche investor by the retailer is given by

$$\xi(Q) \times (K_d - K_a) = e^{-rf\tau} \mu_{\tilde{L}[K_a, K_d]}(Q).$$

It then follows that

$$PR_{CDO}(Q, D) = PR_{NBP}(Q, D) - e^{-rf\tau} \mu_{\tilde{L}[K_a, K_d]}(Q) + e^{-rf\tau} \tilde{L}_{[K_a, K_d]}(Q, D). \quad (20)$$

In parallel with (3), we define

$$W_{CDO}(Q, x) \stackrel{\text{def}}{=} P[PR_{CDO}(Q, D) \leq x]; \quad \pi_{CDO}(Q) \stackrel{\text{def}}{=} E[PR_{CDO}(Q, D)]. \quad (21)$$

The VaR problem with CDO can now be formulated as follows.

$$\begin{aligned} & [\text{VaR-NBP-CDO}] \\ & \min_Q \eta_{CDO} \quad \text{subject to } W_{CDO}(Q, v_0) \leq \eta_{CDO}; \quad \pi_{CDO}(Q) \geq v_1 \end{aligned} \quad (22)$$

In the next section, we express $W_{CDO}(Q, x)$ as a function of Q and establish sufficient conditions under which a stochastic ordering would be present between $PR_{NBP}(Q, D)$ and $PR_{CDO}(Q, D)$. These conditions are also sufficient for the CDO scheme applied to NBP to be effective (or not effective).

3. Sufficient Conditions for CDO Scheme Applied to NBP to Be Effective or Not Effective

The following two theorems in the previous paper are relevant to the discussions in this section. For notational convenience, the following functions are introduced.

$$\xi_+(Q, x) \stackrel{\text{def}}{=} \frac{c_U Q - x}{s}; \quad \xi_-(Q, x) \stackrel{\text{def}}{=} \frac{c_O Q + x}{p - r}, \quad (23)$$

$$\zeta_1(Q, x) = \xi_-(Q, x) + \frac{e^{-rf\tau} \mu_{\tilde{L}[K_a, K_d]}(Q)}{p - r}, \quad (24)$$

$$\zeta_2(Q, x) = \xi_+(Q, x) - \frac{e^{-rf\tau} \mu_{\bar{L}[K_a, K_d]}(Q)}{s}, \quad (25)$$

$$\zeta_3(Q, x) = \frac{x + e^{-rf\tau} \mu_{\bar{L}[K_a, K_d]}(Q) + e^{-rf\tau} K_a}{p - c}, \quad (26)$$

$$\zeta_4(Q, x) = \frac{x + e^{-rf\tau} \mu_{\bar{L}[K_a, K_d]}(Q) + e^{-rf\tau} K_a}{p - c}, \quad (27)$$

$$\zeta_5(Q, x) = \xi_-(Q, x) + \frac{e^{-rf\tau} \mu_{\bar{L}[K_a, K_d]}(Q) - e^{-rf\tau} (K_d - K_a)}{p - r}, \quad (28)$$

$$\zeta_6(Q, x) = \xi_+(Q, x) - \frac{e^{-rf\tau} \mu_{\bar{L}[K_a, K_d]}(Q) - e^{-rf\tau} (K_d - K_a)}{s}. \quad (29)$$

Theorem 3.1 (Isogai, Ohashi and Sumita (2010))

Let $W_{NBP}(Q, x) = P[PR_{NBP}(Q, D) \leq x]$. One then has

$$W_{NBP}(Q, x) = \begin{cases} F_D(\xi_-(Q, x)) + \bar{F}_D(\xi_+(Q, x)) & \text{if } x \leq (p - c)Q \\ 1 & \text{otherwise} \end{cases}.$$

Theorem 3.2 (Isogai, Ohashi and Sumita (2010))

Let $W_{CDO}(Q, x) = P[PR_{CDO}(Q, D) \leq x]$. One then has

$$W_{CDO}(Q, x) = \sum_{i=1}^6 G_i(Q, x),$$

where $G_i(Q, x)$ for $i=1, \dots, 6$, are given as follows.

$$G_1(Q, x) = \begin{cases} 0 & \text{if } x \in (-\infty, x_{1:a}) \\ F_D(\zeta_1(Q, x)) - F_D(Q - \frac{K_a}{c_0}) & \text{if } x \in [x_{1:a}, x_{1:Q}] \\ F_D(Q) - F_D(Q - \frac{K_a}{c_0}) & \text{if } x \in (x_{1:Q}, \infty) \end{cases},$$

where

$$x_{1:Q} = (p - c)Q - e^{-rf\tau} \mu_{\bar{L}[K_a, K_d]}(Q), \quad x_{1:a} = \begin{cases} x_{1:Q} - (p - r) \frac{K_a}{c_0} & \text{if } \frac{K_a}{c_0} \leq Q \\ x_{1:Q} - (p - r)Q & \text{if } Q < \frac{K_a}{c_0} \end{cases}.$$

$$G_2(Q, x) = \begin{cases} 0 & \text{if } x \in (-\infty, x_{2:a}) \\ F_D(Q + \frac{K_a}{c_u}) - F_D(\zeta_2(Q, x)) & \text{if } x \in [x_{2:a}, x_{2:Q}] \\ F_D(Q + \frac{K_a}{c_u}) - F_D(Q) & \text{if } x \in (x_{2:Q}, \infty) \end{cases},$$

where

$$x_{2:Q} = x_{1:Q}, \quad x_{2:a} = x_{1:Q} - \frac{sK_a}{\tilde{c}_U}.$$

$$G_3(Q, x) = \begin{cases} 0 & \text{if } x \in (-\infty, x_{3:d}) \\ F_D(\zeta_3(Q, x)) - F_D(Q - \frac{K_d}{\tilde{c}_O}) & \text{if } x \in [x_{3:d}, x_{3:a}] \\ F_D(Q - \frac{K_a}{\tilde{c}_O}) - F_D(Q - \frac{K_d}{\tilde{c}_O}) & \text{if } x \in (x_{3:a}, \infty) \end{cases},$$

where

$$x_{3:d} = \begin{cases} x_{1:Q} - e^{-r_f \tau} K_a - (p-c) \frac{K_d}{\tilde{c}_O} & \text{if } \frac{K_d}{\tilde{c}_O} \leq Q \\ x_{1:Q} - (p-c)Q - e^{-r_f \tau} K_a & \text{if } Q < \frac{K_d}{\tilde{c}_O} \end{cases}, \quad x_{3:a} = \begin{cases} x_{1:Q} - e^{-r_f \tau} K_a - (p-c) \frac{K_a}{\tilde{c}_O} & \text{if } \frac{K_a}{\tilde{c}_O} \leq Q \\ x_{1:Q} - (p-c)Q - e^{-r_f \tau} K_a & \text{if } Q < \frac{K_a}{\tilde{c}_O} \end{cases}.$$

$$G_4(Q, x) = \begin{cases} 0 & \text{if } x \in (-\infty, x_{4:a}) \\ F_D(\zeta_4(Q, x)) - F_D(Q + \frac{K_a}{\tilde{c}_U}) & \text{if } x \in [x_{4:a}, x_{4:d}] \\ F_D(Q + \frac{K_d}{\tilde{c}_U}) - F_D(Q + \frac{K_a}{\tilde{c}_U}) & \text{if } x \in (x_{4:d}, \infty) \end{cases},$$

where

$$x_{4:a} = x_{1:Q} - \frac{sK_a}{\tilde{c}_U}, \quad x_{4:d} = x_{1:Q} - e^{-r_f \tau} K_a + (p-c) \frac{K_d}{\tilde{c}_U}.$$

$$G_5(Q, x) = \begin{cases} F_D(\zeta_5(Q, x)) & \text{if } x \in (-\infty, x_{5:d}] \\ F_D(Q - \frac{K_d}{\tilde{c}_O}) & \text{if } x \in (x_{5:d}, \infty) \end{cases},$$

where

$$x_{5:d} = \begin{cases} x_{1:Q} + \frac{(c-p)K_d}{\tilde{c}_O} - e^{-r_f \tau} K_a & \text{if } \frac{K_d}{\tilde{c}_O} \leq Q \\ x_{1:Q} - (p-r)Q + e^{-r_f \tau} (K_d - K_a) & \text{if } Q < \frac{K_d}{\tilde{c}_O} \end{cases}.$$

$$G_6(Q, x) = \begin{cases} \bar{F}_D(\zeta_6(Q, x)) & \text{if } x \in (-\infty, x_{6:d}] \\ \bar{F}_D(Q + \frac{K_d}{\tilde{c}_U}) & \text{if } x \in (x_{6:d}, \infty) \end{cases},$$

where

$$x_{6:d} = x_{1:Q} - e^{-r_f \tau} K_a + (p-c) \frac{K_d}{\tilde{c}_U}.$$

The expression of $W_{CDO}(Q, x)$ in Theorem 3.2 is inconvenient to study $W_{CDO}(Q, \nu_0)$ as a function of Q , which is needed to solve the VaR problem with CDO in (22). When $\nu_0 > 0$, $W_{CDO}(Q, \nu_0)$ can be re-expressed as shown in the following theorem. The proof is lengthy, laborious and more or less algebraic, and is omitted here.

Theorem 3.3

Let $x \geq 0$ and define,

$$\begin{aligned} q_1(x) &= \frac{x + e^{-r_f \tau} \mu_{\bar{L}[K_a, K_d]}(Q)}{p-c}; & q_2(x) &= q_1(x) + \left(\frac{K_a}{\tilde{p}-\tilde{c}} - \frac{K_d}{\tilde{c}_U} \right); \\ q_3(x) &= q_1(x) + \frac{sK_a}{(p-c)\tilde{c}_U}; & q_4(x) &= q_1(x) + \frac{K_a(p-r)}{\tilde{c}_O(p-c)}; \\ q_5(x) &= q_1(x) + \left(\frac{K_a}{\tilde{p}-\tilde{c}} + \frac{K_d}{\tilde{c}_O} \right). \end{aligned} \tag{30}$$

The following three statements then hold.

$$(a) \quad \max\{q_1(x), q_2(x)\} < q_3(x) < q_4(x) < q_5(x) \quad \text{and} \quad \max\{q_1(x), q_2(x)\} = \begin{cases} q_2(x) & \text{if } \frac{K_d}{K_a} < \frac{\rho - c + s}{\rho - c} \\ q_1(x) & \text{if } \frac{K_d}{K_a} > \frac{\rho - c + s}{\rho - c} \end{cases}.$$

(b) If $\frac{K_d}{K_a} < \frac{\rho - c + s}{\rho - c}$, then

$$W_{CDO}(Q, x) = \begin{cases} 1 & \text{if } Q \leq q_1(x) \\ F_D(\zeta_1(Q, x)) + \bar{F}_D(\zeta_2(Q, x)) & \text{if } q_1(x) \leq Q \leq q_2(x) \\ F_D(\zeta_1(Q, x)) - F_D(\zeta_2(Q, x)) \\ + F_D(\zeta_4(Q, x)) + \bar{F}_D(\zeta_6(Q, x)) & \text{if } q_2(x) \leq Q \leq q_3(x) \\ F_D(\zeta_1(Q, x)) + \bar{F}_D(\zeta_6(Q, x)) & \text{if } q_3(x) \leq Q \leq q_4(x) \\ F_D(\zeta_3(Q, x)) + \bar{F}_D(\zeta_6(Q, x)) & \text{if } q_4(x) \leq Q \leq q_5(x) \\ F_D(\zeta_5(Q, x)) + \bar{F}_D(\zeta_6(Q, x)) & \text{if } q_5(x) \leq Q \end{cases}.$$

(c) If $\frac{K_d}{K_a} \geq \frac{\rho - c + s}{\rho - c}$, then

$$W_{CDO}(Q, x) = \begin{cases} 1 & \text{if } Q \leq q_2(x) \\ F_D(\zeta_4(Q, x)) + \bar{F}_D(\zeta_6(Q, x)) & \text{if } q_2(x) \leq Q \leq q_1(x) \\ F_D(\zeta_1(Q, x)) - F_D(\zeta_2(Q, x)) \\ + F_D(\zeta_4(Q, x)) + \bar{F}_D(\zeta_6(Q, x)) & \text{if } q_1(x) \leq Q \leq q_3(x) \\ F_D(\zeta_1(Q, x)) + \bar{F}_D(\zeta_6(Q, x)) & \text{if } q_3(x) \leq Q \leq q_4(x) \\ F_D(\zeta_3(Q, x)) + \bar{F}_D(\zeta_6(Q, x)) & \text{if } q_4(x) \leq Q \leq q_5(x) \\ F_D(\zeta_5(Q, x)) + \bar{F}_D(\zeta_6(Q, x)) & \text{if } q_5(x) \leq Q \end{cases}.$$

We are now in a position to establish sufficient conditions under which a stochastic ordering would be present between $PR_{NBP}(Q, D)$ and $PR_{CDO}(Q, D)$. We recall that a nonnegative random variable X is stochastically larger than another nonnegative random variable Y if and only if $\bar{F}_X(x) \geq \bar{F}_Y(x)$ for all $x \geq 0$, where $\bar{F}_X(x) = P[X > x]$ is the survival function of X . This ordering is often denoted by $X \succ_{ST} Y$. In this paper, we extend this concept to arbitrary random variables. More specifically, we define that a random variable X is stochastically larger than another random variable Y on $[0, \infty)$ if and only if $\bar{F}_X(x) \geq \bar{F}_Y(x)$ for all $x \geq 0$. This ordering is denoted by $X \succ_{ST+} Y$.

Theorem 3.4

Let $q_1(x)$, $q_2(x)$ and $q_5(x)$ be as in (30). The following two statements hold true for $x \geq 0$.

(a) If $\frac{K_d}{K_a} < \frac{\rho - c + s}{\rho - c}$ and $q_1(x) \leq Q \leq q_2(x)$, then $PR_{NBP}(Q, D) \succ_{ST+} PR_{CDO}(Q, D)$.

(b) If $Q \geq q_5(x)$ and $\mu_{\mathbb{Z}[K_a, K_d]}(Q) \leq K_d - K_a$, then $PR_{NBP}(Q, D) \prec_{ST+} PR_{CDO}(Q, D)$.

Proof

We note from Theorems 3.1 and 3.3(b) together with (26) and (29) that, for $x \geq 0$,

$$\begin{aligned} & W_{NBP}(Q, x) - W_{CDO}(Q, x) \\ &= \left\{ F_D(\xi_-(Q, x)) - F_D\left(\xi_-(Q, x) + \frac{e^{-rf\tau} \mu_{\tilde{L}[K_a, K_d]}(Q)}{p-r}\right) \right\} \\ & \quad + \left\{ \bar{F}_D(\xi_+(Q, x)) - \bar{F}_D\left(\xi_+(Q, x) - \frac{e^{-rf\tau} \mu_{\tilde{L}[K_a, K_d]}(Q)}{s}\right) \right\}. \end{aligned}$$

Since $F_D(x)$ is monotonically increasing and $\bar{F}_D(x)$ is monotonically decreasing, each of the two terms on the right hand side of the above expression is negative, and $W_{NBP}(Q, v_0) \leq W_{CDO}(Q, v_0)$, proving (a). For part (b), we similarly observe that

$$\begin{aligned} & W_{NBP}(Q, x) - W_{CDO}(Q, x) \\ &= \left\{ F_D(\xi_-(Q, x)) - F_D\left(\xi_-(Q, x) - \frac{e^{-rf\tau}(K_d - K_a) - e^{-rf\tau} \mu_{\tilde{L}[K_a, K_d]}(Q)}{p-r}\right) \right\} \\ & \quad + \left\{ \bar{F}_D(\xi_+(Q, x)) - \bar{F}_D\left(\xi_+(Q, x) + \frac{e^{-rf\tau}(K_d - K_a) - e^{-rf\tau} \mu_{\tilde{L}[K_a, K_d]}(Q)}{s}\right) \right\}. \end{aligned}$$

Under the condition $\mu_{\tilde{L}[K_a, K_d]}(Q) \leq K_d - K_a$, the monotonicity of $F_D(x)$ and that of $\bar{F}_D(x)$ imply that $W_{NBP}(Q, v_0) \geq W_{CDO}(Q, v_0)$, completing the proof. \square

We next show that Theorem 3.4 also provides sufficient conditions for $\eta_{CDO}^{**} \geq \eta_{NBP}^{**}$ or $\eta_{CDO}^{**} \leq \eta_{NBP}^{**}$.

Theorem 3.5

Let $Q_{v_1:L}$ and $Q_{v_1:R}$ be as in (17). Then the following two statements hold.

(a) If $\frac{K_d}{K_a} < \frac{\rho-c+s}{\rho-c}$ and $Q_{v_1:R} \leq q_2(v_0)$, then $\eta_{NBP}^{**} \leq \eta_{CDO}^{**}$.

(b) If $q_5(v_0) \leq Q_{v_1:L}$ and $\mu_{\tilde{L}[K_a, K_d]}(Q) \leq K_d - K_a$ for $\forall Q \in FR(v_1)$, then $\eta_{NBP}^{**} \geq \eta_{CDO}^{**}$.

Proof

Suppose $Q_{v_1:R} \leq q_1(v_0)$. One sees, from Theorem 3.3(b) and (17) together with the conditions in (a), that $W_{CDO}(Q, v_0) = 1$ for $\forall Q \in FR(v_1)$ so that $1 = \eta_{CDO}^{**} \geq \eta_{NBP}^{**}$. If $q_1(v_0) \leq Q_{v_1:R}$, it again follows that $q_1(v_0) \leq Q_{CDO}^{**} \leq Q_{v_1:R} \leq q_2(v_0)$. Consequently, from Theorem 3.4 (a), one has

$$\eta_{CDO}^{**} = W_{CDO}(Q_{CDO}^{**}, v_0) \geq W_{NBP}(Q_{CDO}^{**}, v_0) \geq W_{NBP}(Q_{NBP}^{**}, v_0) = \eta_{NBP}^{**},$$

proving (a). For part (b), if $q_5(v_0) \leq Q_{v_1:L}$ and $\mu_{\tilde{L}[K_a, K_d]}(Q) \leq K_d - K_a$ for $\forall Q \in FR(v_1)$, Theorem 3.4(b) implies that $PR_{NBP}(Q, D) \prec_{ST^+} PR_{CDO}(Q, D)$ for $\forall Q \in FR(v_1)$. It then follows that

$$\eta_{NBP}^{**} = W_{NBP}(Q_{NBP}^{**}, v_0) \geq W_{CDO}(Q_{NBP}^{**}, v_0) \geq W_{CDO}(Q_{CDO}^{**}, v_0) = \eta_{CDO}^{**},$$

completing the proof. \square

4. Analysis of the VaR Problem with Exponential Stochastic Demand

The purpose of this section is to analyze VaR-NBP with the exponentially distributed stochastic demand. This in turn provides a basis for comparing the optimal solution of VaR-NBP with that of VaR-NBP-CDO via numerical examples in Section 5.

Let D be exponentially distributed with p.d.f. $f_D(x)$ defined by

$$f_D(x) = \lambda e^{-\lambda x} U(x), \quad (31)$$

where

$$U(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}. \quad (32)$$

The distribution function and the survival function of D can be written respectively as

$$F_D(x) = (1 - e^{-\lambda x})U(x); \bar{F}_D(x) = 1 - U(x) + e^{-\lambda x}U(x). \quad (33)$$

From (14), one finds that the optimal order quantity Q_{NBP}^* maximizing the expected profit is given by

$$Q_{NBP}^* = \frac{1}{\lambda} \log \left(1 + \frac{c_U}{c_O} \right). \quad (34)$$

From (15) together with (16), it follows that

$$\pi_{NBP}(Q) = -c_O Q - \frac{(c_O + c_U)e^{-\lambda Q}}{\lambda} + \frac{p-r}{\lambda}. \quad (35)$$

The corresponding maximum expected profit is then obtained from (34) and (35) as

$$\pi_{NBP}(Q_{NBP}^*) = -\frac{c_O}{\lambda} \log \left(1 + \frac{c_U}{c_O} \right) + \frac{p-c}{\lambda}. \quad (36)$$

Throughout this section, it is assumed that

$$v_0 \leq (p-c)Q_{v_1:R}. \quad (37)$$

Under this condition, the survival function of $PR_{NBP}(Q, D)$, denoted by $\bar{W}_{NBP}(Q, v_0) = P[PR_{NBP}(Q, D) > v_0] = 1 - W_{NBP}(Q, v_0)$, can be obtained from Theorem 3.1 as

$$\begin{aligned}\bar{W}_{NBP}(Q, v_0) &= e^{-\lambda\xi_-(Q, v_0)} - e^{-\lambda\xi_+(Q, v_0)} \\ &= e^{-\lambda\xi_-(Q, v_0)} \left\{ 1 - e^{-\lambda(\xi_+(Q, v_0) - \xi_-(Q, v_0))} \right\} \\ &= e^{-\lambda\xi_-(Q, v_0)} \left(1 - e^{-\lambda\Delta(Q, v_0)} \right),\end{aligned}\quad (38)$$

where $\Delta(Q, v_0) = \xi_+(Q, v_0) - \xi_-(Q, v_0)$. We are now in a position to prove the next theorem.

Theorem 4.1

Under the condition of (37), the optimal solution of VaR-NBP can be obtained as

$$(Q_{NBP}^{**}, \eta_{NBP}^{**}) = \begin{cases} (Q_{v_1:L}, W_{NBP}(Q_{v_1:L}, v_0)) & \text{if } \hat{Q}_{exp}(v_0) < Q_{v_1:L} \\ (\hat{Q}_{exp}(v_0), W_{NBP}(\hat{Q}_{exp}(v_0), v_0)) & \text{if } Q_{v_1:L} \leq \hat{Q}_{exp}(v_0) \leq Q_{v_1:R} \\ (Q_{v_1:R}, W_{NBP}(Q_{v_1:R}, v_0)) & \text{if } \hat{Q}_{exp}(v_0) > Q_{v_1:R} \end{cases},$$

where
$$\hat{Q}_{exp}(v_0) = \frac{1}{p-c} \left[v_0 + \frac{p-r+s}{\lambda(p-r)s} \log \left\{ 1 + \frac{(p-r+s)(p-c)}{sc_0} \right\} \right].$$

Proof

From (38), one sees that

$$\log \bar{W}_{NBP}(Q, v_0) = -\lambda\xi_-(Q, v_0) + \log(1 - e^{-\lambda\Delta(Q, v_0)}). \quad (39)$$

By differentiating (39) with respect to Q , it follows that

$$\frac{\partial}{\partial Q} \log \bar{W}_{NBP}(Q, v_0) = -\frac{\lambda}{p-r} \left\{ c_0 - \frac{(p-r+s)(p-c)}{s(e^{\lambda\Delta(Q, v_0)} - 1)} \right\}, \quad (40)$$

and

$$\frac{\partial^2}{\partial Q^2} \log \bar{W}_{NBP}(Q, v_0) = -\frac{\lambda^2 e^{\lambda\Delta(Q, v_0)} (p-r+s)^2 (p-c)^2}{(e^{\lambda\Delta(Q, v_0)} - 1)^2 (p-r)^2 s^2} < 0. \quad (41)$$

Hence, $\log \bar{W}_{NBP}(Q, v_0)$ is strictly concave in Q and so is $\bar{W}_{NBP}(Q, v_0)$. Accordingly,

$\bar{W}_{NBP}(Q, v_0)$ has the unique maximum at $\hat{Q}_{exp}(v_0)$, satisfying $\left. \frac{\partial}{\partial Q} \log \bar{W}_{NBP}(Q, v_0) \right|_{Q=\hat{Q}_{exp}(v_0)} = 0$.

This then implies that $W_{NBP}(Q, v_0)$ takes the unique minimum at $\hat{Q}_{exp}(v_0)$. From (40), it can be readily seen that

$$\hat{Q}_{exp}(v_0) = \frac{1}{p-c} \left[v_0 + \frac{p-r+s}{\lambda(p-r)s} \log \left\{ 1 + \frac{(p-r+s)(p-c)}{sc_0} \right\} \right].$$

If $\hat{Q}_{\text{exp}}(v_0) < Q_{v_1:L}$, then $W_{NBP}(Q, v_0)$ is monotonically increasing in $Q \in FR(v_1)$ and

$(Q_{NBP}^{**}, \eta_{NBP}^{**}) = (Q_{v_1:L}, W_{NBP}(Q_{v_1:L}, v_0))$. If $Q_{v_1:L} \leq \hat{Q}_{\text{exp}}(v_0) \leq Q_{v_1:R}$, $W_{NBP}(Q, v_0)$ clearly takes the minimum value at $\hat{Q}_{\text{exp}}(v_0)$ and therefore $(Q_{NBP}^{**}, \eta_{NBP}^{**}) = (\hat{Q}_{\text{exp}}(v_0), W_{NBP}(\hat{Q}_{\text{exp}}(v_0), v_0))$.

Finally, when $\hat{Q}_{\text{exp}}(v_0) > Q_{v_1:R}$, $W_{NBP}(Q, v_0)$ is monotonically decreasing in $Q \in FR(v_1)$ so that $(Q_{NBP}^{**}, \eta_{NBP}^{**}) = (Q_{v_1:R}, W_{NBP}(Q_{v_1:R}, v_0))$, completing the proof. \square

5. Numerical Results

In this section, we illustrate Theorem 3.5 through numerical examples. Furthermore, in order to explore the potential of the CDO scheme applied to NBP in a more general context, the optimal solution for VaR-NBP and that for VaR-NBP-CDO would be compared by altering the underline parameters v_1 , v_0 and ρ . The basic set of the parameter values is provided in Table 5.1, which would be employed throughout this section unless specified otherwise.

Table 5.1 Basic Set of Parameter Values

ρ	the unit sales price	3
c	the unit procurement cost	1
r	the unit residual value	0.1
s	the unit opportunity cost	0.5
μ_D	the mean of the demand	5000
r_f	the risk free rate	0.0001
K_a	the attachment point	500
K_d	the detachment point	600, 1000, 3000, 5000

The expected loss $\mu_{f,NBP}(Q)$ and the expected profit $\pi(Q)$ are first plotted in Figures 5.1 and 5.2, respectively. The former is strictly convex and the latter is strictly concave, as expected.

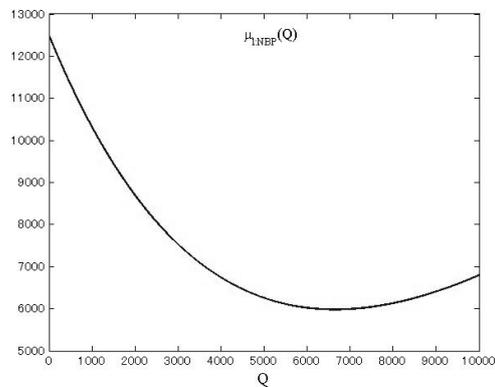


Figure 5.1 Expected Loss $\mu_{f,NBP}(Q)$

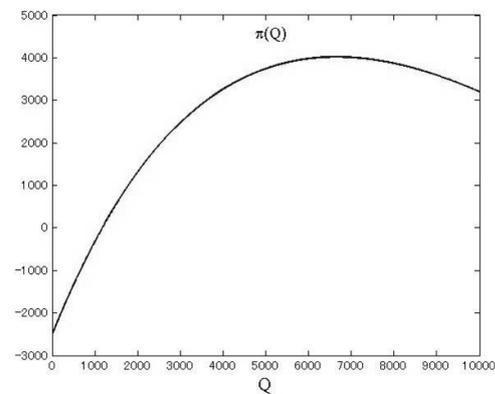


Figure 5.2 Expected Profit $\pi(Q)$

Figure 5.3 illustrates Theorem 5.3 (a) with $\nu_1 = 4000$ and $\nu_0 = 14200$. One sees from Table 5.1 that

$$\frac{K_d}{K_a} = 1.2 < 1.25 = \frac{\rho - c + s}{\rho - c} \quad \text{and} \quad Q_{\nu_1:R} = 7111 \leq 7158 = q_2(\nu_0),$$

and the conditions are satisfied. We observe that

$$(Q_{NBP}^{**}, \eta_{NBP}^{**}) = (Q_{\nu_1:R}, W_{NBP}(Q_{\nu_1:R}, \nu_0)) = (7111, 0.997)$$

and

$$(Q_{CDO}^{**}, \eta_{CDO}^{**}) = (Q_{\nu_1:R}, W_{CDO}(Q_{\nu_1:R}, \nu_0)) = (7111, 1.000)$$

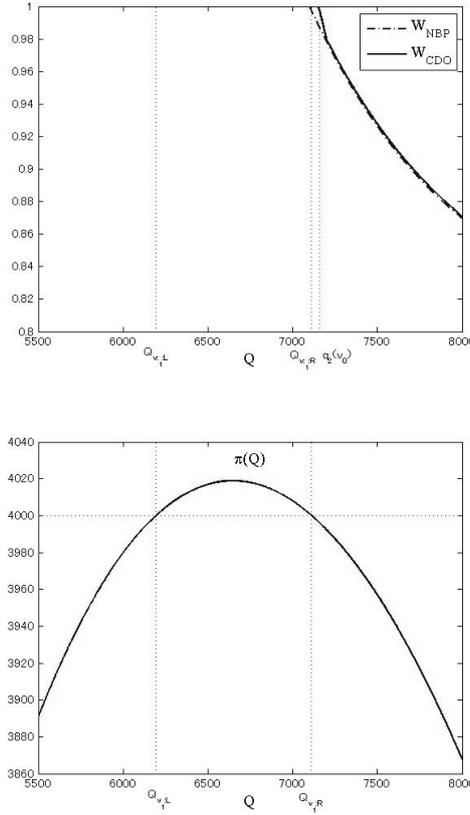


Figure 5.3 Theorem 3.5 (a)

Theorem 5.3 (b) is illustrated numerically in Figure 5.4. We note that

$$q_3(\nu_0) = 5578 \leq 6195 = Q_{\nu_1:L}.$$

Figure 5.3 shows $\mu_{\mathbb{Z}[K_a, K_d]}(Q)$ as a function of Q , demonstrating $\mu_{\mathbb{Z}[K_a, K_d]}(Q) \leq K_d - K_a$ so that the conditions for Theorem 5.3 (b) are satisfied. For this case, we observe that

$$(Q_{NBP}^{**}, \eta_{NBP}^{**}) = (Q_{\nu_1:L}, W_{NBP}(Q_{\nu_1:L}, \nu_0)) = (6195, 0.411)$$

and

$$(Q_{CDO}^{**}, \eta_{CDO}^{**}) = (Q_{\nu_1:L}, W_{CDO}(Q_{\nu_1:L}, \nu_0)) = (6195, 0.392).$$

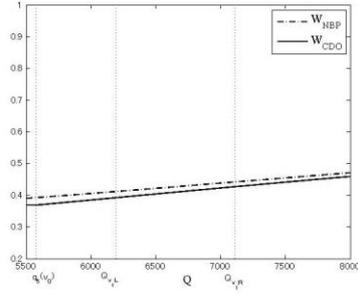


Figure 5.4 Theorem 3.5(b)

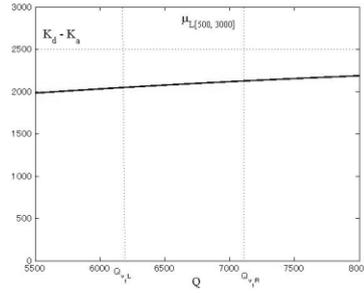
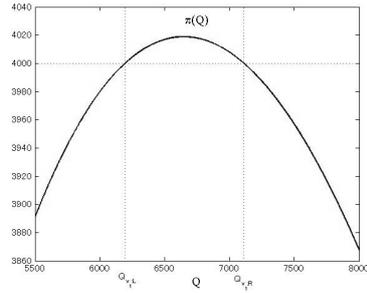


Figure 5.5 $\mu_{L[K_d, K_d]}(Q)$

In Figure 5.6, $W_{NBP}(Q, 2000)$ and $W_{CDO}(Q, 2000)$ are plotted along with $\pi(Q)$, where v_1 is varied from 3000 and 3500 to 4000, while $K_d = 3000$ is fixed. One sees that the feasible region $FR(v_1)$ becomes narrower as the threshold, v_1 , of the expected profit increases. Accordingly, both η_{NBP}^{**} and η_{CDO}^{**} become worse and increase as v_1 increases. It is worth noting that the CDO approach is effective only when v_1 becomes sufficiently large.

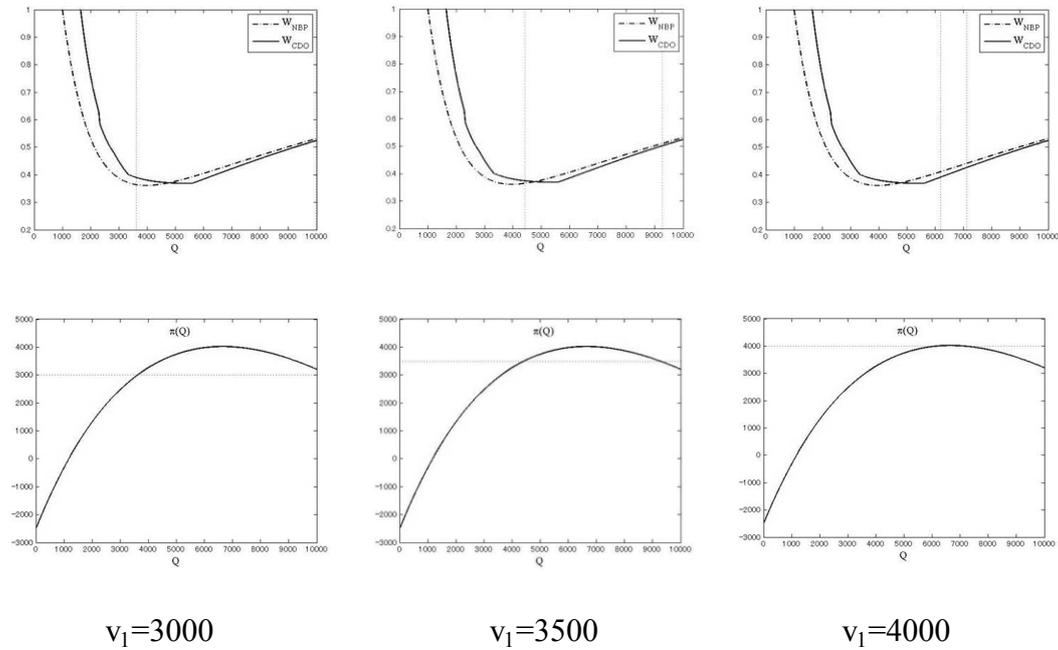


Figure 5.6 $\pi(Q)$ and $W_{NBP}(Q, 2000)$ vs. $W_{CDO}(Q, 2000)$ [$K_d = 3000$]

Similarly to Figure 5.6, $W_{NBP}(Q, \nu_0)$, $W_{CDO}(Q, \nu_0)$ and $\pi(Q)$ are depicted in Figure 5.7, where ν_0 is varied from 1000 and 1500 to 2000, while ν_1 is fixed at 3500 with $K_d = 3000$. It can be observed that it becomes more difficult to control the profit as the threshold ν_0 becomes larger. While $\eta_{NBP}^{**} > \eta_{CDO}^{**}$ for $\nu_0 = 1000, 1500$, this inequality is reversed and the CDO approach becomes ineffective for $\nu_0 = 2000$.

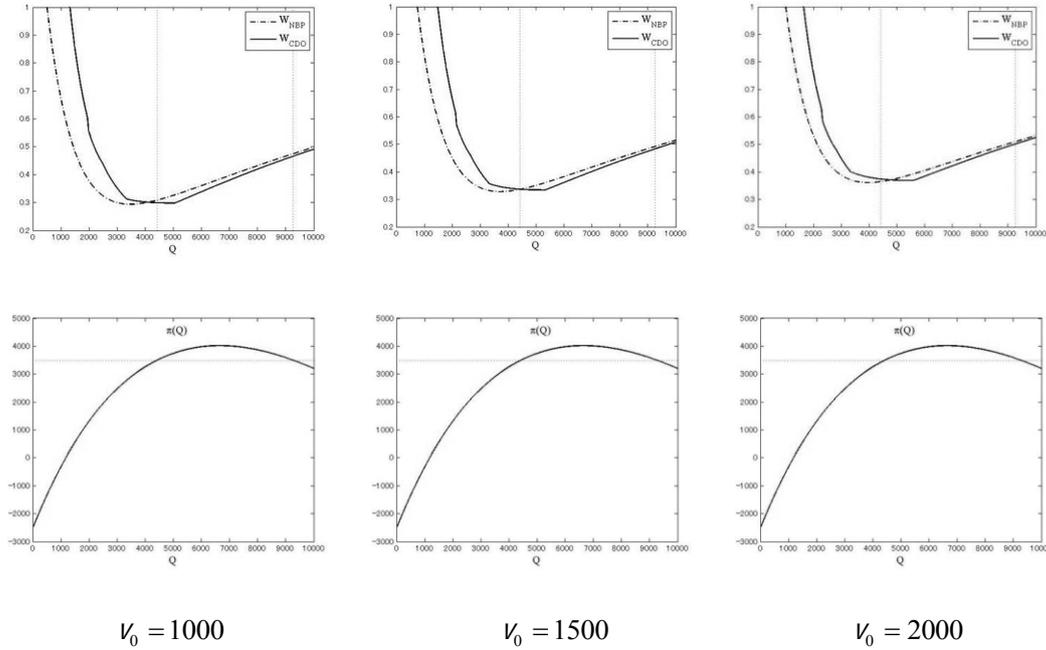


Figure 5.7 $\pi(Q)$ and $W_{NBP}(Q, \nu_0)$ vs. $W_{CDO}(Q, \nu_0)$ [$K_d = 3000$, $\nu_1 = 3500$]

In order to observe the impact of the price on the optimal solutions more closely, Figure 5.8 depicts η_{NBP}^{**} and η_{CDO}^{**} as functions of ρ , where $\nu_0 = 2000$, $\nu_1 = 3500$ and $K_d = 3000$ are fixed. The CDO approach becomes effective when ρ becomes sufficiently large.

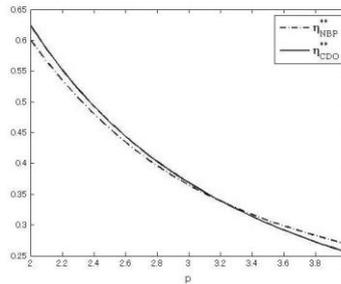


Figure 5.8 η_{NBP}^{**} vs. η_{CDO}^{**} [$K_d = 3000$, $\nu_0 = 2000$, $\nu_1 = 3500$]

Finally, Figure 5.9 illustrates how η_{NBP}^{**} and η_{CDO}^{**} are impacted when (ν_0, ν_1) and K_d are changed, where the white areas represent the regions in which the CDO approach is effective. It can be observed that the CDO approach can be effective only when ν_1 is sufficiently large for $K_d = 5000$. The area in which the CDO approach

performs better shifts toward the lower side of v_0 and becomes larger as K_d decreases.

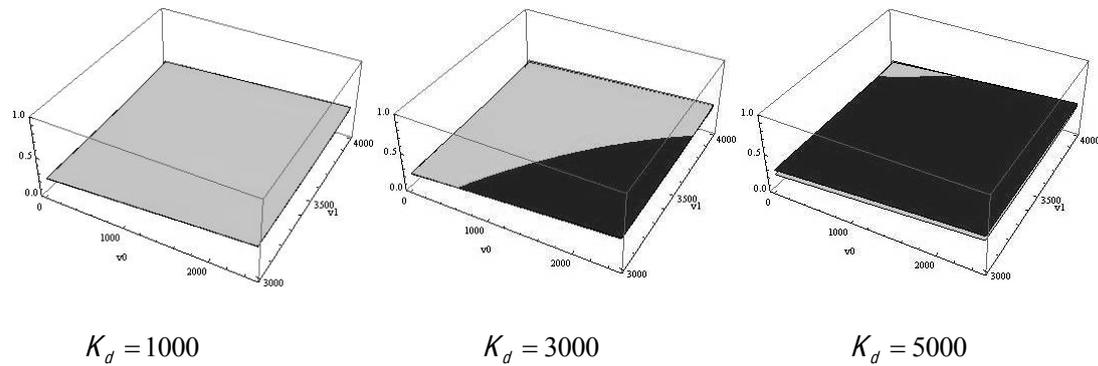


Figure 5.9 η_{NBP}^{**} vs. η_{CDO}^{**} as (v_0, v_1) and K_d Change

6. Concluding Remarks

As a sequel to the original paper by the same authors, Isogai, Ohashi and Sumita (2010), this paper examines the effect of the CDO scheme applied to the classical NBP. The distribution function $W_{CDO}(Q, x)$ of $PR_{CDO}(Q, D)$ is re-expressed as a function of Q . Furthermore, sufficient conditions are established under which a stochastic ordering would be present between $PR_{NBP}(Q, D)$ and $PR_{CDO}(Q, D)$. It is shown that these conditions are also sufficient to assure $\eta_{CDO}^{**} \geq \eta_{NBP}^{**}$ (or $\eta_{CDO}^{**} \leq \eta_{NBP}^{**}$). The VaR problem of NBP without CDO is analyzed in detail for the case of the exponentially distributed demand, deriving Q_{NBP}^{**} and η_{NBP}^{**} explicitly.

Extensive numerical experiments reveal that the overall effect of CDO is present when the underlying risk for the opportunity loss is rather large. More specifically, assuming that the stochastic demand D is exponentially distributed, the CDO approach could become effective if

- (i) the expected profit should be held above a high level;
- (ii) the probability of having a huge loss should be contained;
- (iii) the price is very high; and
- (iv) the detachment point K_d should be held relatively low.

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