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by

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Structural Analysis of Optimal Investment Strategy with Budget Constraints for Project Management: Real Option Approach

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Abstract

A Dynamic Programming approach is proposed for managing projects with uncertain risks. As in Huchzermeier and Loch [3], this real option approach does not require the underlying asset of the project to be traded in a market since alternative options of the project do not have to be replicated as financial options. In Huchzermeier and Loch [3], the necessary investments are treated as exogenous functions of time and the decision is limited to whether or not the option of terminating the projects should be exercised at each time stage. This paper extends their framework substantially by incorporating the optimal investment strategy with budget constraints explicitly. Structural properties of the optimal investment strategy are investigated in detail, establishing certain monotonicity properties of the optimal project value and the optimal investment amount, as well as convexity properties of the optimal project value. Some numerical results are also presented.

Keywords: Dynamic Programming, Project Management, Real Option

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1 Introduction

In evaluation of project values, one of the most prevalent methods is DCF (Discounted Cash Flow) where the future cash flow generated by a project, when it is completed as planned, is discounted to the present value using the capital cost as a discounting parameter. This discounted future cash flow is compared with the initial investment amount, yielding NPV (Net Present Value) of the project as the difference of the two.If $NPV \geq 0$, the project would be carried out, while it would be terminated when $NPV < 0$. Klammer [7] reported that only 15 % of U.S. companies employed DCF in 1959, but the percentage was increased to 57 % in 1970. Today almost all of U.S. companies use DCF for evaluation of project values, see e.g. Yamamoto and Kariya [12]. This approach, however, cannot explicitly incorporate uncertainty arising from development of the project.

In order to overcome this difficulty, ROAP (Real Option Approach) has been recently drawing much attention of researchers and practitioners. Following Yamamoto and Kariya [12], ROAP is defined in this paper as below:

(1.1) Real option is the right of the management to explore alternative options in a management environment with high uncertainty.

(1.2) ROV (Real Option Value) is the portion of the present project value representing the value of having alternative options.

(1.3) A method to evaluate ROV is called ROAP.

Typically alternative options include termination, deferral, expansion, contract, time to build, transfer, shutdown and restart, cancellation, market entry (Yamamoto and Kariya [12]), improvement (Huchzermeier and Loch [3]), exchange option (Lee and Paxson [9]), and growth option (Loch and Bode-Greuel [10]). ROAP is superior to DCF when the degree of uncertainty is higher and/or various alternative options are available. In addition, ROAP is more useful than DCF when the initial investment needed to carry out the project is larger than the discounted future cash flow. In this case, one has $NPV < 0$ and the project would be terminated if DCF is employed. In reality, however, such a project itself may be traded in the market. The potential project value of this sort can be captured by ROAP, but not by DCF. Typical projects for which ROAP is more attractive than DCF include: mining natural resource projects (Cortazer, Shwartz, and Casassus [2]), gas and electric projects (Yamamoto and Kariya [12]), infrastructure

development projects (Yamamoto and Kariya [12]), IT projects (Benarch [1], Kumer [8]), pharmaceuticals R&D project (Kellogg and Charnes [5], Trang, Takezawa, and Takezawa [11]), tree harvesting problems (Insley [4]), and lease projects (Kenyon and Tompaidis [6]).

In ROAP, it is widely observed that alternative options for the project management are treated as financial options, and the underlying uncertainty is evaluated accordingly. In some literature, this specific approach is called ROAP. In this paper, however, we stick to the original definition of (1.1) through (1.3), and the above approach is called " the risk-neutral" ROAP. As discussed in Kellogg and Charnes [5], the risk-neutral ROAP has the advantage of providing substantial flexibility in incorporating a variety of alternative options, and of eliminating the laborious evaluation of the capital cost which is replaced by the risk-free rate. However, the major draw-back of the risk-neutral ROAP can be found in that it requires the underlying asset of the project to be traded in a market.Otherwise alternative options of the project cannot be replicated as financial options, destructing the foundation of this approach. Among the projects previously mentioned as those preferring ROAP, only mining natural resource projects and gas and electric projects satisfy this condition. In order to eliminate the risk-neutral requirement, a paper by Huchzermeier and Loch [3] employs a DP (Dynamic Programming) approach, where the success probability of the project for each time stage does not have to be risk-neutral but arbitrary, and the capital cost is replaced by a risk-free rate as the discounting parameter. In the paper [3], however, investment costs are treated as functions of time, exogenous to the underlying desicion structure.

The purpose of this paper is to develop a DP-based ROAP for determining the optimal investment strategy with budget constraints so as to maximize the expected present project value. Here investment costs are incorporated explicitly as a part of strategic decisions within the model, thereby extending the framework of [3] substantially. Probability of success at time t is treated as an increasing function of the investment amount to be decided at time t. A similar framework can be found in Kellogg and Charnes [5] but such probabilities are assumed to be constant there. Salvage values are also incorporated when the option to terminate the project is exercised or the project was forced to stop due to failure. Furthermore, such values are expressed as increasing functions of the estimated value of the project outcome at the occurrence of the stoppage. Thang, Takezawa, and Takezawa [11] treated option salvage values, but they were assumed to

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be constants. Salvage values upon failure are also incorporated in a similar manner. It will be shown that the optimal project value with options, V^*, is always larger than that without options, $\hat{V}^*$, so that the optimal real option value $ROV^* = V^* - \hat{V}^*$ is always nonnegative. Furthermore, both $\hat{V}^*$ and V^* increase as the level of uncertainty involved in the successful completion of the project decreases. Certain convexity properties are also shown with some additional conditions. For the optimal investment amount x^*, similar monotonicity properties can be present under more restrictive conditions.

The structure of this paper is as follows. In Section 2, a project management model based on ROAP is formally introduced. Two associated DP problems are formulated in Section 3. It is shown that under certain conditions the unique optimal investment strategy exists for each of the DP problems. Sections 4 and 5 are devoted to establish structural properties of the optimal project value V^* and the optimal investment amount x^*. Certain convexity proerties of V^* are derived in Section 6. Numerical results are exhibited in Section 7, demonstrating the monotonicity and convexity properties of V^* as well as the monotonicity properties of x^*. Basic properties of certain concave functions are given in Appendix, which will play a key role throughout the paper.

2   Model Description

We consider a project management problem over T periods. Let S_0 be an estimeted value of the project outcome at time $t = 0$. This estimated value may increase or decrease as the project evolves. In managing this project toward the end of period T, one has an option to terminate the project at the beginning of each period t, $1 \leq t \leq T$, with some salvage value. The decision criterion for this option will soon become clear. If it is decided to continue the project, the investment amount for this period should be determined subject to a budget constraint. The project may be carried out successfully to the next period $t + 1$ or may fail. The success probability may depend on the investment amount. When successful, the estimated value of the project outcome increases by a factor of u with probability p or decreases by a factor of d with probability $1 - p$, where $0 < d < 1 < u$ and $0 \leq p \leq 1$. The former case is called an upward success and the latter is called a downward success. When the project fails in period t, the project is forced to stop with certain salvage value, which is different from the salvage value under the option of termination.

The decision structure described above can be expressed as a modified binary tree in the following manner. Suppose that period t is completed with k upward and $t-k$ downward successes at time t, and we are at the beginning of period $t+1$. This state is denoted by (t,k), $0 \le k \le t$. Let $S_{t,k}$ be the estimated value of the project outcome at state (t,k) so that

$$S_{t,k} = S_0 u^k d^{t-k}, \quad 0 \le k \le t. \tag{2.1}$$

Similarly, we introduce:

$$\hat{V}_{t,k} \quad : \quad \textit{the expected value of the project at state } (t,k) \tag{2.2}$$
$$\textit{without option for termination,}$$

and

$$V_{t,k} \quad : \quad \textit{the expected value of the project at state } (t,k) \tag{2.3}$$
$$\textit{with option for termination.}$$

When the decision is made to terminate the project at state (t,k), we denote the salvage value by $V_{A:t+1,k}$, i.e.

$$V_{A:t+1,k} \quad : \quad \textit{the salvage value of the project} \tag{2.4}$$
$$\textit{when it is decided to terminate it at state } (t,k).$$

The corresponding state is denoted by $(A : t+1, k)$.

If the project is continued, the investment amount $x_{t+1,k}$ is determined subject to a budget constraint not to exceed c_{t+1}. How to determine $x_{t+1,k}$ will be discussed in Section 3. Let $\beta(x)$ be the probability that the project can be continued successfully for one period given that the investment amount for the period is x. Throughout the paper, we assume that, for $x \ge 0$,

$$\beta(0) = 0, \quad 0 \le \beta(x) \le 1, \quad \beta'(x) > 0, \quad \beta''(x) < 0, \tag{2.5}$$

where $\beta'(x) = \frac{d}{dx}\beta(x)$ and $\beta''(x) = \frac{d^2}{dx^2}\beta(x)$, which reflects the fact that the success probability increases as the investment amount increases with the effect of diminishing return. The project experiences an upward success with probability $\beta(x_{t+1,k}) \cdot p$. In this case, state moves from (t,k) to $(t+1, k+1)$. With probability $\beta(x_{t+1,k}) \cdot (1-p)$, a

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downward success is realized and state moves from (t, k) to $(t + 1, k)$.

The project may be forced to stop because of failure with probability $1 - \beta(x_{t+1,k})$. In this case, the associated salvage value is denoted by $V_{F:t+1,k}$, i.e.

$$V_{F:t+1,k} \quad : \quad the\ salvage\ value\ of\ the\ project \qquad (2.6)$$
$$when\ it\ is\ forced\ to\ stop$$
$$because\ of\ failure\ starting\ from\ state\ (t, k).$$

The corresponding state is denoted by $(F : t+1, k)$. The structure of these state transitions is depicted in Figure 2.1.
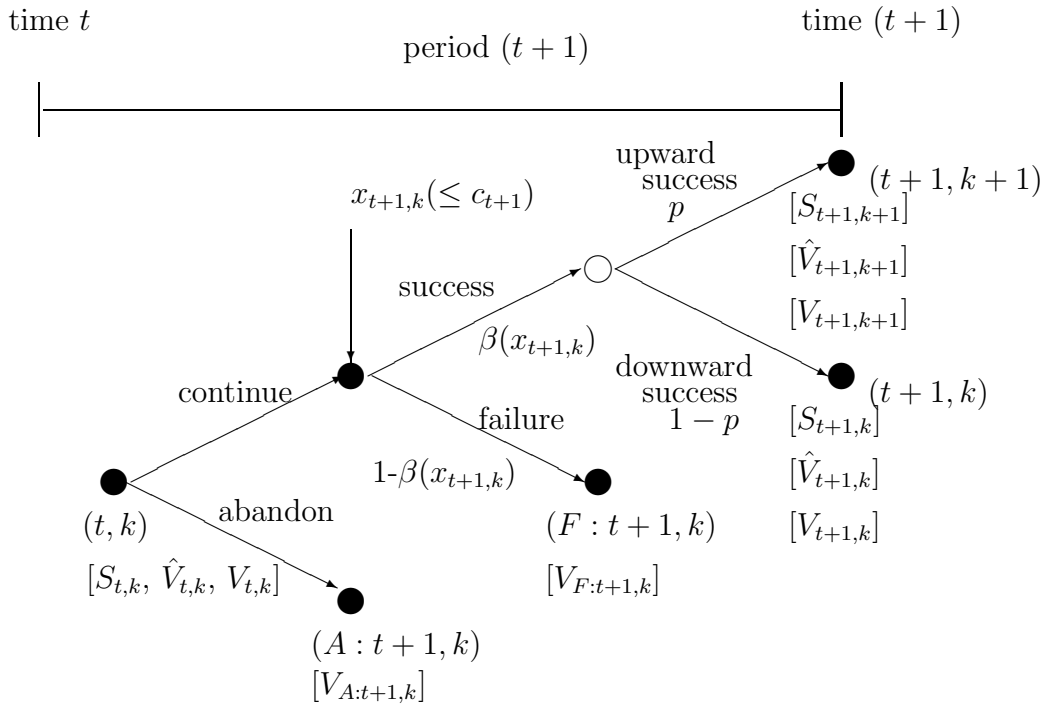

Figure 2.1 Modified Binary Tree Stucture for One Period

3 Formulation of Optimal Investment Policy: Dynamic Programming Approach

From the modified binary tree in Figure 2.1, if the project continues with success to the final period T, the project value $V_{T,k}$ at time T with k upward successes and $T - k$ downward successes can be written as

$$\hat{V}_{T,k} = V_{T,k} = S_{T,k} - X_T = S_0 u^k d^{T-k} - X_T, \quad 0 \le k \le T, \qquad (3.1)$$

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where X_T is the operational cost needed for generating the cash flow from the completed project. For the two salvation values $V_{F:t+1,k}$ upon failure and $V_{A:t+1,k}$ due to decision to terminate, we assume that both are functions of $S_{t,k}$ and define

$$V_{F:t+1,k} = W_F(S_{t,k}); \quad V_{A:t+1,k} = W_A(S_{t,k}). \tag{3.2}$$

It is natural to assume that $W_F(\cdot)$ and $W_A(\cdot)$ are zero without investment and are strictly increasing and concave in x, i.e,

$$W_F(0) = 0, \quad W_F'(x) \geq 0, \quad W_F''(x) \leq 0; \tag{3.3}$$
$$W_A(0) = 0, \quad W_A'(x) \geq 0, \quad W_A''(x) \leq 0.$$

When no option for terminating the project is available, the corresponding expected project value given the investment amount $\hat{x}_{t+1,k}$ satisfies the following backward recursive formula.

$$\hat{V}_{t,k} = \left\{ \beta(\hat{x}_{t+1,k}) \left(p\hat{V}_{t+1,k+1} + (1-p)\hat{V}_{t+1,k} \right) \right. \tag{3.4}$$
$$\left. + \left(1 - \beta(\hat{x}_{t+1,k}) \right) V_{F:t+1,k} \right\} e^{-r} - \hat{x}_{t+1,k}.$$

Accordingly the optimal investment strategy without options should be determined so as to maximize $\hat{V}_{0,0}$. This problem can be formulated as the following DP problem. For notational convenience, let $G(\lambda, A, B)$ be defined by

$$G(\lambda, A, B) = \lambda A + (1 - \lambda)B \tag{3.5}$$

where $0 \leq \lambda \leq 1$.

$$[\text{DP-}\hat{V}]$$

$$max_{[\hat{x}_{t,k}]} \quad \hat{V}_{0,0} \tag{3.6}$$

subject to

$$\hat{V}_{t,k} = G\left(\beta(\hat{x}_{t+1,k}), G(p, \hat{V}_{t+1,k+1}, \hat{V}_{t+1,k}), V_{F:t+1,k} \right) e^{-r} - \hat{x}_{t+1,k} \tag{3.7}$$
$$with \quad 0 \leq \hat{x}_{t+1,k} \leq c_{t+1}, \quad 0 \leq k \leq t, \quad 0 \leq t \leq T-1;$$

$$S_{t,k} = S_0 u^k d^{t-k}, \quad 0 \leq k \leq t, \quad 0 \leq t \leq T; \tag{3.8}$$
$$\hat{V}_{T,k} = S_{T,k} - X_T, \quad 0 \leq k \leq T; \ and \tag{3.9}$$
$$V_{F:t+1,k} = W_F(S_{t,k}), \quad 0 \leq k \leq t, \quad 0 \leq t \leq T-1. \tag{3.10}$$

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Since $G(p, \hat{V}_{t+1,k+1}, \hat{V}_{t+1,k})$ is the expectd project value with success while $V_{F:t+1,k}$ is the salvage value upon failure, we assume that

$$G(p, \hat{V}_{t+1,k+1}, \hat{V}_{t+1,k}) > V_{F:t+1,k}. \tag{3.11}$$

It should be noted that $DP - \hat{V}$ can be solved recursively by finding

$$\hat{V}_{t,k}^* = \hat{f}_{t,k}(\hat{x}_{t+1,k}^*) = \max_{0 \le x \le c_{t+1}} \hat{f}_{t,k}(x); \tag{3.12}$$
$$\hat{f}_{t,k}(x) = G(\beta(x), \hat{A}_{t,k}, \hat{B}_{t,k}) - x$$

where

$$\hat{A}_{t,k} = G(p, \hat{V}_{t+1,k+1}^*, \hat{V}_{t+1,k}^*)e^{-r}; \quad \hat{B}_{t,k} = V_{F:t+1,k}e^{-r} \tag{3.13}$$

for $0 \le k \le t$ and $0 \le t \le T - 1$, starting with

$$\hat{V}_{T,k}^* = \hat{V}_{T,k} = S_{T,k} - X_T, \quad 0 \le k \le T. \tag{3.14}$$

With option for termination, the expected value of the project has to be compared with the salvage value of the project for termination at the beginning of each period. Hence the backward recursive formula for the expected project value $V_{t,k}$ given the investment amount $x_{t+1,k}$ should be

$$V_{t,k} = \max\Big\{G(\beta(x_{t+1,k}), G(p, V_{t+1,k+1}, V_{t+1,k}), V_{F:t+1,k})e^{-r} - x_{t+1,k}, V_{A:t+1,k}\Big\}. \tag{3.15}$$

The DP formulation then becomes:

$$[\,\text{DP-V}\,]$$

$$max_{[x_{t,k}]} \quad V_{0,0} \tag{3.16}$$

subject to

$$V_{t,k} = \max\Big\{G(\beta(x_{t+1,k}), G(p, V_{t+1,k+1}, V_{t+1,k}), V_{F:t+1,k})e^{-r} - x_{t+1,k}, V_{A:t+1,k}\Big\}, \tag{3.17}$$
$$with \quad 0 \le x_{t+1,k} \le c_{t+1}, \quad 0 \le k \le t, \quad 0 \le t \le T - 1;$$

$$S_{t,k} = S_0 u^k d^{t-k}, \quad 0 \le k \le t, \quad 0 \le t \le T; \tag{3.18}$$
$$V_{T,k} = S_{T,k} - X_T, \quad 0 \le k \le T; \tag{3.19}$$
$$V_{F:t+1,k} = W_F(S_{t,k}), \quad 0 \le k \le t, \quad 0 \le t \le T - 1; \; and \tag{3.20}$$
$$V_{A:t+1,k} = W_A(S_{t,k}), \quad 0 \le k \le t, \quad 0 \le t \le T - 1. \tag{3.21}$$

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Similar to (3.11), one has

$$G(p, V_{t+1,k+1}, V_{t+1,k}) > V_{F:t+1,k}. \tag{3.22}$$

In parallel with (3.12) through (3.14), one sees that $DP - V$ can be solved recursively by finding

$$V_{t,k}^* = \max\{f_{t,k}(x_{t+1,k}^*), V_{A:t+1,k}\}; \tag{3.23}$$

$$f_{t,k}(x_{t+1,k}^*) = \max_{0 \le x \le c_{t+1}} f_{t,k}(x); \quad f_{t,k}(x) = G(\beta(x), A_{t,k}, B_{t,k}) - x$$

where

$$A_{t,k} = G(p, V_{t+1,k+1}^*, V_{t+1,k}^*)e^{-r}; \quad B_{t,k} = V_{F:t+1,k}e^{-r} \tag{3.24}$$

for $0 \le k \le t$ and $0 \le t \le T - 1$, starting with

$$V_{T,k}^* = V_{T,k} = S_{T,k} - X_T, \quad 0 \le k \le T. \tag{3.25}$$

For the option value at state (t, k), we define

$$ROV_{t,k}^* = V_{t,k}^* - \hat{V}_{t,k}^*. \tag{3.26}$$

Of particular interest is to find the option value $ROV_{0,0}^*$ and the associated optimal investment strategy $(x_{t,k}^*)$ at the start of the project for the optimal investment policy problem.

We next show that, when it is decided to continue the project, the optimal investment amount can be determined uniquely under (2.5) for both $DP - \hat{V}$ and $DP - V$, which facilitates the necessary DP computation substantially. We note from (3.11), (3.13), (3.22), and (3.24) that

$$\hat{A}_{t,k} > \hat{B}_{t,k}; \quad A_{t,k} > B_{t,k}; \quad \hat{B}_{t,k} = B_{t,k}. \tag{3.27}$$

Theorem 3.1 *Let*

$$\hat{z}_{t+1,k}^* = \arg\max_{x \ge 0} \hat{f}_{t,k}(x); \quad z_{t+1,k}^* = \arg\max_{x \ge 0} f_{t,k}(x) \tag{3.28}$$

where $\hat{f}_{t,k}(x)$ and $f_{t,k}(x)$ are given in (3.12) and (3.23) respectively. Suppose that the success probability function $\beta(x)$ satisfies (2.5). Then, whenever it is decided to continue the project at (t, k), the following statements hold.
a) $\hat{z}_{t+1,k}^*$ and $z_{t+1,k}^*$ are determined uniquely.
b) Both $DP - \hat{V}$ and $DP - V$ have the unique optimal investment amounts

$$\hat{x}_{t+1,k}^* = \min\{\hat{z}_{t+1,k}^*, c_{t+1}\}; \quad x_{t+1,k}^* = \min\{z_{t+1,k}^*, c_{t+1}\} \tag{3.29}$$

for all k, $0 \le k \le t$, and all t, $0 \le t \le T - 1$.

Proof For part a), from (2.5), (3.12) and (3.27), it can be readily seen that $\hat{f}_{t,k}(x)$ is strictly concave. For $DP - \hat{V}$, if $\hat{f}'_{t,k}(0) < 0$, then $\hat{f}'_{t,k}(x) < 0$, since $\hat{f}''_{t,k}(x) < 0$, for all $x \geq 0$ and the unique maximum value of $\hat{f}_{t,k}(x)$ for $x \geq 0$ is attained at $\hat{z}^* = 0$. Otherwise, from (2.5), (3.12), strictly concavity of $\hat{f}_{t,k}(x)$ and $\lim_{x \to \infty} \hat{f}_{t,k}(x) = -\infty$, $\hat{f}_{t,k}(x)$ has the unique maximum point $\hat{z}^*_{t+1,k}$ determined by

$$\beta'(\hat{z}^*_{t+1,k}) = (\hat{A}_{t,k} - B_{t,k})^{-1}. \tag{3.30}$$

Under the assumption that the project is continued at (t, k), similar arguments can be repeated for $DP - V$ with $f_{t,k}(x)$ of (3.23), proving part a). Part b) follows from part a) and Lemma A.1 a), completing the proof. $\qquad\square$

Example 3.2 Let $\beta(x) = \frac{bx}{1+bx}$ with $b > 0$. Then $\beta(x)$ satisfies the conditions in (2.5). One has from Theorem 3.1,

$$\hat{x}^*_{t+1,k} \;=\; \min\left\{\frac{-1 + \sqrt{b(\hat{A}_{t,k} - B_{t,k})}}{b}, c_{t+1}\right\} \quad for \quad DP - \hat{V} \tag{3.31}$$

and

$$x^*_{t+1,k} \;=\; \min\left\{\frac{-1 + \sqrt{b(A_{t,k} - B_{t,k})}}{b}, c_{t+1}\right\} \quad for \quad DP - V. \tag{3.32}$$

This example will be used in Section 6 for numerical exploration.

4 Structural Properties of $\hat{V}^*$ and V^*

In this section, we derive various monotonicity properties of the expected project values $\hat{V}^*_{t,k}$ and $V^*_{t,k}$. Furthermore, it is shown that the option value at state (t, k), denoted by $ROV^*_{t,k}$ in (3.26), is always nonnegative. We first show that both $\hat{V}^*_{t,k}$ and $V^*_{t,k}$ increase as k increases, i.e. the more upward successes the project experiences, the larger the expected project value is.

Theorem 4.1 Let $0 \leq k \leq t - 1$ for $1 \leq t \leq T$. Then:

a) $\hat{V}^*_{t,k+1} > \hat{V}^*_{t,k}$

b) $V^*_{t,k+1} > V^*_{t,k}$

Proof We prove part a) by backward induction. For $t = T$, one sees from (3.1) that

$$\hat{V}^*_{T,k+1} - \hat{V}^*_{T,k} = S_0 u^k d^{T-k-1}(u - d) > 0,$$

since $0 < d < 1 < u$. Suppose that part a) holds for $t + 1$ and consider the case of t. Let $\hat{A}_{t,k}$ and $B_{t,k}$ be as in (3.13) and (3.27) respectively. From the induction hypothesis, one has

$$\hat{A}_{t,k+1} - \hat{A}_{t,k} = \{p(\hat{V}^*_{t+1,k+2} - \hat{V}^*_{t+1,k+1}) + (1 - p)(\hat{V}^*_{t+1,k+1} - \hat{V}^*_{t+1,k})\}e^{-r} \quad (4.1)$$

$$> 0$$

i.e.

$$\hat{A}_{t,k+1} > \hat{A}_{t,k}. \quad (4.2)$$

From (2.1) and $0 < d < 1 < u$, it can be readily seen that

$$S_{t,k+1} = S_0 u^{k+1} d^{t-(k+1)} > S_0 u^k d^{t-k} = S_{t,k}. \quad (4.3)$$

From the monotonicity of $W_F(x)$ in (3.3) together with (3.2), (3.13) and (3.27), it then follows that

$$B_{t,k+1} = W_F(S_{t,k+1})e^{-r} > W_F(S_{t,k})e^{-r} = B_{t,k}. \quad (4.4)$$

Therefore, part a) follows from (4.2), (4.4) and Lemma A.2. Part b) can be shown similarly. □

Parameters p, u and d represent the level of uncertainty involved in the successful completion of the project. The next theorem shows monotonicity of $\hat{V}^*_{t,k}$ and $V^*_{t,k}$ in terms of these parameters, i.e. the expected project value increases as the level of uncertainty decreases.

Theorem 4.2 *For $0 \leq k \leq t$ and $0 \leq t \leq T - 1$, the following statements hold.*

a) $\hat{V}^*_{t,k}$ *is strictly increasing in p, u, and d.*

b) $V^*_{t,k}$ *is nondecreasing in p.*

c) $V^*_{t,k}$ *is strictly increasing in u and d.*

Proof As before, we prove the monotnicity of $\hat{V}^*_{t,k}$ in p by backward induction. Proofs for other cases are similar and omitted here. Let $p_1 > p_2$. For $i = 1, 2$, we define

$$\hat{V}^*_{t,k}(\hat{x}^*_{i:t+1,k}, p_i) = \hat{f}_{t,k}(\hat{x}^*_{i:t+1,k}, p_i) = \max_{0 \leq x \leq c_{t+1}} \hat{f}_{t,k}(x, p_i) \qquad (4.5)$$

and

$$\hat{f}_{t,k}(\hat{z}^*_{i:t+1,k}, p_i) = \max_{x \geq 0} \hat{f}_{t,k}(x, p_i) \qquad (4.6)$$

where, from (3.12) and (3.13),

$$\hat{f}_{t,k}(x, p_i) = G(\beta(x), \hat{A}_{t,k}(p_i), B_{t,k}) - x \qquad (4.7)$$

and

$$\hat{A}_{t,k}(p_i) = G(p_i, \hat{V}^*_{t+1,k+1}(\hat{x}^*_{i:t+2,k+1}, p_i), \hat{V}^*_{t+1,k}(\hat{x}^*_{i:t+2,k}, p_i))e^{-r}. \qquad (4.8)$$

Since $\hat{V}_{T,k} = S_0 u^k d^{T-k}$ is independent of p, it can be readily seen that one has $\hat{A}_{T-1,k}(p_1) > \hat{A}_{T-1,k}(p_2)$. It then follows from (4.5) through (4.8) and Lemma A.5 a) that

$$\hat{V}^*_{T-1,k}(\hat{x}^*_{1:T,k}, p_1) > \hat{V}^*_{T-1,k}(\hat{x}^*_{2:T,k}, p_2). \qquad (4.9)$$

Suppose that (4.9) holds true with $t+1$ replacing $T-1$ and consider the case of t. From the induction hypothesis, it can be readily seen that $\hat{A}_{t,k}(p_1) > \hat{A}_{t,k}(p_2)$. Applying Lemma A.5 a) again, it follows that (4.9) holds with t in place of $T-1$, proving part a) for the case of p. □

The next theorem states that the project values $\hat{V}^*_{t,k}$ and $V^*_{t,k}$ are nondecreasing in c_τ for each (t, k) satisfying $t+1 \leq \tau$. Proof is similar to that of Theorem 4.2 except that Lemmas A.1 b) and A.2 are used instead of Lemma A.5 a), and is omitted here.

Theorem 4.3 $\hat{V}^*_{t,k}$ and $V^*_{t,k}$ are nondecreasing in c_τ, $t+1 \leq \tau$.

In parallel with the proof of Theorem 4.1, the monotonicity property of $\hat{V}^*_{t,k}$ and $V^*_{t,k}$ also holds true in terms of the success probability $\beta(x)$, as shown in the next theorem. Proof is again omitted.

Theorem 4.4 Let $\beta_1(x)$ and $\beta_2(x)$ be as in (2.5) satisfying $\beta_1(x) \leq \beta_2(x)$ for all $x \geq 0$. For $i = 1, 2$, let $\hat{V}^*_{i:t,k}$ and $V^*_{i:t,k}$ be the corresponding project values. Then $\hat{V}^*_{1:t,k} \leq \hat{V}^*_{2:t,k}$ and $V^*_{1:t,k} \leq V^*_{2:t,k}$.

The next corollary is immediate from Theorem 4.4.

Corollary 4.5 *Let* $\beta(x,b) = \frac{bx}{1+bx}$, *as in Example 3.2. Then* $\hat{V}_{t,k}^*$ *is strictly increasing and* $V_{t,k}^*$ *is nondecreasing in* b.

We next show that the optimal project value with options, $V_{t,k}^*$, is larger than that without options, $\hat{V}_{t,k}^*$ so that the optimal option value $ROV_{t,k}^*$ is nonnegative for all t,k.

Theorem 4.6 *For* $0 \le k \le t$ *and* $0 \le t \le T$, $ROV_{t,k}^* = V_{t,k}^* - \hat{V}_{t,k}^* \ge 0$.

Proof One sees from (3.1) that $V_{T,k} = \hat{V}_{T,k}$ and $ROV_{T,k}^* = 0$. Suppose $ROV_{t+1,k}^* \ge 0$ and consider the case of t. By the induction hypothesis, one has $V_{t+1,k} \ge \hat{V}_{t+1,k}$. From (3.13) and (3.24), this in turn implies that

$$A_{t,k} > \hat{A}_{t,k}. \tag{4.10}$$

It then follows from (3.27), (4.10) and Lemma A.2 that $V_{t,k}^* \ge \hat{V}_{t,k}^*$ completing the proof. $\square$

5 Structural Properties of $\hat{x}^*$ and x^*

In this section, we establish similar monotonicity properties for the optimal investment amount without option for termination, $\hat{x}_{t,k}^*$, and that with option $x_{t,k}^*$. Our first theorem below shows that having the option for termination prevides an incentive to invest more because the risk involved can be controlled better.

Theorem 5.1 *If it is decided to continue the project at state* (t,k), *then* $x_{t+1,k}^* \ge \hat{x}_{t+1,k}^*$ *for* $0 \le k \le t$, $0 \le t \le T-1$.

Proof The theorem follows from (4.10) together with (3.12), (3.23), (3.27) and Lemma A.3. $\square$

We next derive the monotonicity property of $\hat{x}_{t,k}^*$ and $x_{t,k}^*$ in k, i.e. the optimal investment amount increases as more upward succeses are experienced. In contrast with Theorem 4.1 for $\hat{V}_{t,k}^*$ and $V_{t,k}^*$, the monotonicity property of $\hat{x}_{t,k}^*$ and $x_{t,k}^*$ involves certain subtlety and the following two assumptions are needed.

Assumption 5.2

a) $W_F(x) = \alpha_F x, \; \alpha_F > 0$

b) $pu + (1 - p)d > 1$

For notational convenience, the first difference of a double sequence $(a_{t,k})_{t,k=0}^{\infty}$ with respect to k is denoted by

$$\Delta_k a_{t,k} = a_{t,k} - a_{t,k-1}, \quad k \geq 1. \tag{5.1}$$

A preliminary lemma is needed.

Lemma 5.3 *If $\hat{z}_{t+2,k-1}^* < \hat{z}_{t+2,k}^*$, then*

$$\frac{\Delta_k \hat{x}_{t+2,k}^*}{\hat{A}_{t+1,k} - B_{t+1,k}} \leq \Delta_k \beta(\hat{x}_{t+2,k}^*). \tag{5.2}$$

Proof From (3.29), one sees that $\hat{x}_{t+2,k}^* = \hat{z}_{t+2,k}^*$ or c_{t+2} and $\hat{x}_{t+2,k-1}^* = \hat{z}_{t+2,k-1}^*$ or c_{t+2}. Accordingly three cases should be examined separately.

Case 1: $\hat{z}_{t+2,k-1}^* < \hat{z}_{t+2,k}^* \leq c_{t+2}$

From strict concavity of $\beta(x)$ together with (3.30), it can be seen that

$$\beta'(\hat{z}_{t+2,k}^*)\Delta_k \hat{z}_{t+2,k}^* = \frac{\Delta_k \hat{z}_{t+2,k}^*}{\hat{A}_{t+1,k} - B_{t+1,k}} < \Delta_k \beta(\hat{z}_{t+2,k}^*), \tag{5.3}$$

and the lemma follows since $\hat{z}_{t+2,k}^* = \hat{x}_{t+2,k}^*$ for this case.

Case 2: $\hat{z}_{t+2,k-1}^* < c_{t+2} < \hat{z}_{t+2,k}^*$

By the mean value theorem and strict concavity of $\beta(x)$, there exists $z \in (\hat{z}_{t+2,k-1}^*, c_{t+2})$ such that

$$\beta'(\hat{z}_{t+2,k}^*) < \beta'(c_{t+2}) < \beta'(z) = \frac{\beta(c_{t+2}) - \beta(\hat{z}_{t+2,k-1}^*)}{c_{t+2} - \hat{z}_{t+2,k-1}^*}. \tag{5.4}$$

Since $\Delta_k(\hat{x}_{t+2,k}^*) = c_{t+2} - \hat{z}_{t+2,k-1}^*$ and $\Delta_k \beta(\hat{x}_{t+2,k}^*) = \beta(c_{t+2}) - \beta(\hat{z}_{t+2,k-1}^*)$ for this case, (5.3) together with (3.30) implies (5.2).

Case 3: $c_{t+2} \leq \hat{z}_{t+2,k-1}^* < \hat{z}_{t+2,k}^*$

For this case, one has $\hat{x}_{t+2,k}^* = \hat{x}_{t+2,k-1}^* = c_{t+2}$. It then follows that $\Delta_k \hat{x}_{t+2,k}^* = 0$ and $\Delta_k \beta(\hat{x}_{t+2,k}^*) = 0$ so that (5.2) is satisfied with equality, proving the lemma. $\quad\square$

 We are now in a position to prove the following theorem.

Theorem 5.4 *Under Assumption 5.2, one has for $0 \leq k \leq t$, $0 \leq t \leq T - 1$:*

a) $\min\{\Delta_k \hat{A}_{t,k}, \Delta_k A_{t,k}\} > \Delta_k B_{t,k}.$

b) $\hat{x}_{t+1,k}^*$ *and* $x_{t+1,k}^*$ *are nondecreasing in* k.

Proof We prove $\Delta_k \hat{A}_{t,k} > \Delta_k B_{t,k}$ and the monotonicity of $\hat{x}^*_{t+1,k}$ in k by backward induction. Proof for $\Delta_k A_{t,k} > \Delta_k B_{t,k}$ and the monotonicity of $x^*_{t+1,k}$ in k is similar and omitted. Since $\hat{A}_{t,k} > B_{t,k}$ from (3.27), one has, in particular, $\hat{A}_{T-1,k} > B_{T-1,k}$. From (2.1), (3.13), (3.14), and Assumption 5.2 a), it then follows that

$$
\begin{aligned}
(\hat{A}_{T-1,k} - B_{T-1,k})e^r &= G(p, \hat{V}_{T,k+1}, \hat{V}_{T,k}) - W_F(S_{T-1,k}) \\
&= p(S_{T,k+1} - X_T) + (1-p)(S_{T,k} - X_T) - \alpha_F S_{T-1,k} > 0,
\end{aligned}
$$

i.e.

$$
(\hat{A}_{T-1,k} - B_{T-1,k})e^r = S_{T-1,k}\{pu + (1-p)d - \alpha_F\} - X_T > 0. \tag{5.5}
$$

This, in turn, implies that

$$
pu + (1-p)d - \alpha_F > 0. \tag{5.6}
$$

By taking the first difference of both sides of (5.5) with respect to k, one then sees that

$$
\begin{aligned}
(\Delta_k \hat{A}_{T-1,k} - \Delta_k \hat{B}_{T-1,k})e^r &= \{pu + (1-p)d - \alpha_F\}\Delta_k S_{T-1,k} \\
&= \{pu + (1-p)d - \alpha_F\}\left(1 - \frac{d}{u}\right) S_{T-1,k}.
\end{aligned}
$$

Since $0 < d < u$, from (5.6), this then leads to

$$
\Delta_k \hat{A}_{T-1,k} > \Delta_k B_{T-1,k}. \tag{5.7}
$$

Inequality (5.7) and Lemma A.3 imply that

$$
\Delta_k \hat{x}^*_{T,k} > 0. \tag{5.8}
$$

Suppose that (5.7) and (5.8) hold true when $T-1$ is replaced by $t+1$ and T is replaced by $t + 2$ respectively. We first show that

$$
\Delta_k \hat{V}^*_{t+1,k} > \Delta_k B_{t+1,k}. \tag{5.9}
$$

Noting $\Delta_k[a_k b_k] = b_k \Delta_k a_k + a_{k-1} \Delta_k b_k$, one sees from (3.5) and (3.12) that

$$
\begin{aligned}
&\Delta_k \hat{V}^*_{t+1,k} - \Delta_k B_{t+1,k} \\
&= \Delta_k[\beta(\hat{x}^*_{t+2,k})\hat{A}_{t+1,k} + (1 - \beta(\hat{x}^*_{t+2,k}))B_{t+1,k} - \hat{x}^*_{t+2,k}] - \Delta_k B_{t+1,k} \\
&= [\hat{A}_{t+1,k} - B_{t+1,k}]\Delta_k \beta(\hat{x}^*_{t+2,k}) - \Delta_k \hat{x}^*_{t+2,k} \\
&\quad + \beta(\hat{x}^*_{t+2,k-1})[\Delta_k \hat{A}_{t+1,k} - \Delta_k B_{t+1,k}].
\end{aligned}
$$

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By the induction hypothesis of (5.7) with $T - 1$ replaced by $t + 1$, the last term in the above expression is positive so that

$$\Delta_k \hat{V}^*_{t+1,k} - \Delta_k B_{t+1,k} > [\hat{A}_{t+1,k} - B_{t+1,k}]\Delta_k \beta(\hat{x}^*_{t+2,k}) - \Delta_k \hat{x}^*_{t+2,k}. \tag{5.10}$$

From the induction hypothesis, one has $\Delta_k \hat{x}^*_{t+2,k} > 0$. Employing Lemma 5.3 in (5.10) then yields (5.9).

Finally, one sees from (3.5) and (3.13) that

$$
\begin{aligned}
(\Delta_k \hat{A}_{t,k} - \Delta_k B_{t,k})e^r &= \{G(p, \Delta_k \hat{V}_{t+1,k+1}, \Delta_k \hat{V}_{t+1,k}) - \Delta_k B_{t,k}\}e^r \\
&\geq \{G(p, \Delta_k B_{t+1,k+1}, \Delta_k B_{t+1,k}) - \Delta_k B_{t,k}\}e^r,
\end{aligned}
$$

where (5.9) is used to derive the last inequality. From Assumption 5.2 a) together with (3.2), it can be readily seen that

$$\Delta_k B_{t+1,k+1} = \alpha_F u \left(1 - \frac{d}{u}\right) S_{t,k}e^{-r}; \tag{5.11}$$

$$\Delta_k B_{t+1,k} = \alpha_F d \left(1 - \frac{d}{u}\right) S_{t,k}e^{-r};$$

$$\Delta_k B_{t,k} = \alpha_F \left(1 - \frac{d}{u}\right) S_{t,k}e^{-r}.$$

Substituting (5.11) into the last inequality above, it follows that

$$(\Delta_k \hat{A}_{t,k} - \Delta_k B_{t,k})e^r \geq \alpha_F \left(1 - \frac{d}{u}\right) S_{t,k}\{pu + (1 - p)d - 1\}.$$

Hence one concludes from Assumption 5.2 b) that

$$\Delta_k \hat{A}_{t,k} > \Delta_k B_{t,k}. \tag{5.12}$$

Using (5.12), Lemma A.3 then yields $\Delta_k \hat{x}^*_{t+1,k} > 0$, completing the proof. $\qquad \square$

For a function $\xi(z)$ with $z_1 > z_2$, we define

$$\Delta_z \xi(z) = \xi(z_1) - \xi(z_2). \tag{5.13}$$

With this notation, the monotonicity properties of $\hat{x}^*_{t,k}$ and $x^*_{t,k}$ with respect to u and d can be shown as in the theorem below. Proof is almost identical to that of Theorem 5.4, and is omitted.

Theorem 5.5

a) *Let $u_1 > u_2$ and suppose Assumption 5.2 is satisfied where $u = u_2$. Then*

$$\min\{\Delta_u \hat{A}_{t,k}(u), \Delta_u A_{t,k}(u)\} > \Delta_u B_{t,k}(u)$$

and both $\hat{x}_{t,k}^*$ and $x_{t,k}^*$ are strictly increasing in u.

b) *Let $d_1 > d_2$ and suppose Assumption 5.2 is satisfied where $d = d_2$. Then*

$$\min\{\Delta_d \hat{A}_{t,k}(d), \Delta_d A_{t,k}(d)\} > \Delta_d B_{t,k}(d)$$

and both $\hat{x}_{t,k}^*$ and $x_{t,k}^*$ are strictly increasing in d.

The monotonicity properties of $\hat{x}_{t,k}^*$ and $x_{t,k}^*$ with respect to p hold true without Assumption 5.2 as we prove next.

Theorem 5.6 *Let $p_1 > p_2$. Then*

$$\min\{\Delta_p \hat{A}_{t,k}(p), \Delta_p A_{t,k}(p)\} > \Delta_p B_{t,k}(p)$$

and both $\hat{x}_{t,k}^*$ and $x_{t,k}^*$ are strictly increasing in p.

Proof   We first note from (3.2) and (3.24) that $B_{t,k} = W_F(S_{t,k})$ which is independent of p. Hence $\Delta_p B_{t,k} = 0$. On the other hand, it can be readily seen from (3.13), (3.24) and Theorem 4.2 that $\Delta_p \hat{A}_{t,k} > 0 = \Delta_p B_{t,k}$ and $\Delta_p A_{t,k} > 0 = \Delta_p B_{t,k}$. Lemma A.5 b) then implies that $\Delta_p \hat{x}_{t+1,k}^*(p) > 0$ and $\Delta_p x_{t+1,k}^*(p) > 0$, i.e. both $\hat{x}_{t+1,k}^*$ and $x_{t+1,k}^*$ are strictly increasing in p, proving the theorem.   □

The next theorem is immediate from Lemma A.1 b).

Theorem 5.7 *$\hat{x}_{t,k}^*$ and $x_{t,k}^*$ are nondecreasing in c_t.*

6   Convexity Properties of $\hat{V}^*$ and V^*

For a sequence $(a_{t,k})_{k=0}^\infty$, the second difference is denoted by

$$\Delta_k^2 a_{t,k} \;=\; \Delta_k a_{t,k} - \Delta_k a_{t,k-1} \tag{6.1}$$

$$=\; a_{t,k} - 2a_{t,k-1} + a_{t,k-2}, \quad k \geq 2.$$

We say that $(a_{t,k})_{k=0}^\infty$ is discrete convex if $\Delta_k^2 a_{t,k} \geq 0$. The convexity is strict if $\Delta_k^2 a_{t,k} > 0$. It should be noted that

$$(a_{t,k})_{k=0}^\infty \text{ is discrete convex} \iff \frac{a_{t,k} + a_{t,k+2}}{2} \geq a_{t,k+1}, \; k = 0, 1, 2, \cdots, \tag{6.2}$$

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where the strict convexity is present if and only if the inequality above is strict. In this section, various convexity properties of $\hat{V}_{t,k}$ and $V_{t,k}$ are established. We first show that both $\hat{V}^*_{t,k}$ and $V^*_{t,k}$ are discrete convex in k when both $W_F(x)$ and $W_A(x)$ are linear.

Theorem 6.1 *Under Assmption 5.2, suppose $W_A(x) = \alpha_A x$ where $\alpha_A > 0$. Then the following statements hold.*

a) $\hat{V}^*_{t,k}$ *is strictly discrete convex in k.*

b) $V^*_{t,k}$ *is strictly discrete convex in k.*

Proof We prove part a) by backward induction. For $t = T$, one has

$$
\begin{aligned}
\Delta^2_k \hat{V}_{T,k} &= \Delta^2_k \{ S_0 u^k d^{T-k} - X_T \} &&(6.3)\\
&= S_0 u^{k-2} d^{T-k} (u-d)^2 \\
&> 0.
\end{aligned}
$$

Suppose part a) holds true at time $t + 1$, i.e.

$$
\Delta^2_k \hat{V}^*_{t+1,k} > 0. \tag{6.4}
$$

From (3.5), (3.13) and (6.4), one sees that

$$
\begin{aligned}
\Delta^2_k \hat{A}_{t,k} &= \Delta^2_k G(p, \hat{V}^*_{t+1,k+1}, \hat{V}^*_{t+1,k}) e^{-r} &&(6.5)\\
&= G(p, \Delta^2_k \hat{V}^*_{t+1,k+1}, \Delta^2_k \hat{V}^*_{t+1,k}) e^{-r} \\
&> 0.
\end{aligned}
$$

On the other hand, it can be seen that

$$
\Delta^2_k B_{t,k} = \alpha_F S_0 u^{k-2} d^{t-k} (u-d)^2 e^{-r} > 0. \tag{6.6}
$$

Consequently both $\hat{A}_{t,k}$ and $B_{t,k}$ are strictly discrete convex in k and the condition (A.8) of Lemma A.4 is satisfied from (6.2). We also see from (4.1), (4.4) and Theorem 5.4 a) that other conditions of Lemma A.4 are satisfied. Part a) then follows from Lemma A.4 by recognizing $\hat{V}_{t,k} = \hat{f}_{t,k}(\hat{x}^*_{t+1,k})$ as given in (3.12). Part b) can be shown similarly by noting that if $\gamma_{i:k}$ are discrete convex with respect to k for $1 \le i \le N$, then so is $\max_{1 \le i \le N} \{ \gamma_{i:k} \}$. $\qquad\square$

Convexity properties of $\hat{V}_{t,k}$ and $V_{t,k}$ with respect to u and d also hold true. Proof is similar to that of Theorem 6.1 and is omitted.

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Theorem 6.2 *Under Assmption 5.2, suppose $W_A(x) = \alpha_A x$ where $\alpha_A > 0$. Then the following statements hold.*

a) *$\hat{V}_{t,k}$ is strictly convex in both u and d.*

b) *$V_{t,k}$ is strictly convex in both u and d.*

7 Numerical Results

In this section, we present numerical results to demonstrate the monotonicity properties of V and x, and the convexity properties of V derived in the previous three sections. Throughout this section, it is assumed that the success probability $\beta(x)$ is of the form given in Example 3.2, and two salvage value functions $W_A(x)$ for option to terminate and $W_F(x)$ for failure are both linear. More specifically, we define:

$$\beta(x) = \frac{bx}{1 + bx}, \quad b > 0; \tag{7.1}$$

$$W_A(x) = \alpha_A x, \quad \alpha_A > 0; \ and \tag{7.2}$$

$$W_F(x) = \alpha_F x, \quad 0 < \alpha_F < \alpha_A. \tag{7.3}$$

As a basic model, we adopt parameter values specified in the table below. The monetary unit is one million yen and the time unit is one year. These parameter values are assumed throughout this section unless specified otherwise.

Symbol	Definition	Value
S_0	the estimated future cash flow of the project outcome	3,000
T	the planning horizon	5
p	the upward probability given success	0.85
r	the risk-free interest rate	0.1
u	the increasing rate given an upward success	1.4
d	the decreasing rate given a downward success	0.7
X_T	the operational cost	100
b	the success probability parameter in (7.1)†	0.015
α_A	the salvage value parameter in (7.2)	0.3
α_F	the salvage value parameter in (7.3)	0.1
c_{t+1}	the budget constraint at period $t(= 0, 1, 2, 3, 4)$	300

Remark 7.1† *The value of b is set so that $\beta(x)$ would likely to lie between 0.6 and 0.95 for the basic model.*

Table 7.1 summarizes computational results for the basic model, while Table 7.2 exhibits those for a model similar to the basic model except that p is reduced from 0.85 to

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0.6. We note that $\hat{V}^*_{t,k+1} \geq \hat{V}^*_{t,k}$ and $V^*_{t,k+1} \geq V^*_{t,k}$ as they should be from Theorem 4.1. This monotonicity is also observed for the investment amounts, i.e. $\hat{x}^*_{t,k+1} \geq \hat{x}^*_{t,k}$ and $x^*_{t,k+1} \geq x^*_{t,k}$. Furthermore, we note that $\hat{V}^*_{t,k+2} - \hat{V}^*_{t,k+1} \geq \hat{V}^*_{t,k+1} - \hat{V}^*_{t,k}$ and $V^*_{t,k+2} - V^*_{t,k+1} \geq V^*_{t,k+1} - V^*_{t,k}$ as shown in Section 6.

Comparing Table 7.1 with Table 7.2, one finds that both $\hat{V}_{0,0}$ and $V^*_{0,0}$ decrease as p decreases, which is expected form Theorem 4.2. Furthermore it can be seen that the option value in Table 7.2 with p=0.6 is 349.8, while that for the basic model with p=0.85 is 0. This suggests that the risk potential of the project increases as p decreases, and the option for terminating the project becomes a viable alternative. The increase of the risk potential can also be observed in the fact that the investment amounts with $p = 0.6$ are uniformly smaller than those with $p = 0.85$, i.e. $x^*_{t,k}$ with p=0.6 are less than $x^*_{t,k}$ with p=0.85 for all $0 \leq k \leq t$, $0 \leq t \leq T - 1 = 4$.

In order to explore monotonicity properties of $\hat{V}^*_{t,k}$ and $V^*_{t,k}$, further, numerical experiments are conducted by varying parameters p (Figure 7.3), u (Figure 7.4), d (Figure 7.5), and b (Figure 7.6). As we already observed, the risk potential increases as p decreases. From Example 3.2, one has $\frac{\partial}{\partial b}\beta > 0$ so that the success probability β increases as b increases, which in turn leads to reduction of the risk potential. It is clear that increasing u or d results in larger $\hat{V}^*_{0,0}$ and $V^*_{0,0}$ and the risk potential decreases. Figures 7.7 through 7.9 exhibit similar monotonicity properties for $\hat{x}^*$ and x^*. Assumption 5.2 b) is satisfied for $p > 0.58$ and $u > 1.18$. However the monotonicity of $\hat{x}^*$ and x^* is observed outside these ranges.

Numerical results presented in this section suggest that the option value $ROV^*_{0,0}$ increases as the risk potential increases. Theoritical proof for this conjecture is difficult and is being attempted.

c_{t+1}	t	k	DP-V $V^*_{t,k}$	$x^*_{t+1,k}$	DP-$\hat{V}$ $\hat{V}^*_{t,k}$	$\hat{x}^*_{t+1,k}$	$ROV^*_{t,k}$
300	0	0	1737.9	300	1737.9	300	0
300	1	1	2969.7	300	2969.7	300	0
300	1	0	1079.1	243.5	1079.1	243.5	0
300	2	2	4732.7	300	4732.7	300	0
300	2	1	2003	300	2003	300	0
300	2	0	678.9	190.8	678.9	190.8	0
300	3	3	7254.7	300	7254.7	300	0
300	3	2	3338.9	300	3338.9	300	0
300	3	1	1382.7	282.4	1382.7	282.4	0
300	3	0	468.9	158.3	468.9	158.3	0
300	4	4	10864.6	300	10864.6	300	0
300	4	3	5245.3	300	5245.3	300	0
300	4	2	2435.6	300	2435.6	300	0
300	4	1	1038.7	246.1	1038.7	246.1	0
300	4	0	391.8	147.6	391.8	147.6	0

Table 7.1 $p = 0.85$ and $c_t = 300$ for all t

c_{t+1}	t	k	DP-V $V^*_{t,k}$	$x^*_{t+1,k}$	DP-$\hat{V}$ $\hat{V}^*_{t,k}$	$\hat{x}^*_{t+1,k}$	$ROV^*_{t,k}$
300	0	0	900	149.5	550.2	136.3	349.8
300	1	1	1370.7	257	1370.7	257	0
300	1	0	630	126.7	401.5	118.8	228.5
300	2	2	2868.3	300	2868.3	300	0
300	2	1	1088.9	234.2	1088.9	234.2	0
300	2	0	441	112.1	316.5	110.6	124.5
300	3	3	5330.4	300	5330.4	300	0
300	3	2	2376.7	300	2376.7	300	0
300	3	1	921.8	221.4	921.8	221.4	0
300	3	0	308.7	113.8	287.3	113.8	21.4
300	4	4	9371.5	300	9371.5	300	0
300	4	3	4498.7	300	4498.7	300	0
300	4	2	2062.3	300	2062.3	300	0
300	4	1	861.3	220.7	861.3	220.7	0
300	4	0	314.8	129	314.8	129	0

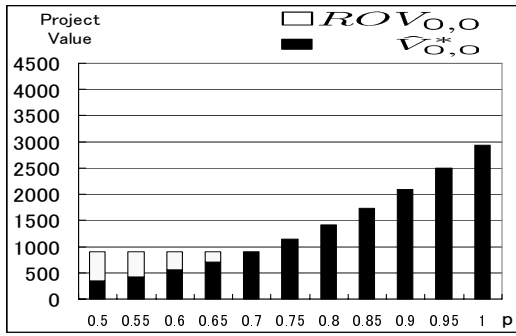Table 7.2 $p = 0.6$ and $c_t = 300$ for all t

Figure 7.3 Structural Properties about p of $\hat{V}_{0,0}^*$, $V_{0,0}^*$, and $ROV_{0,0}^*$
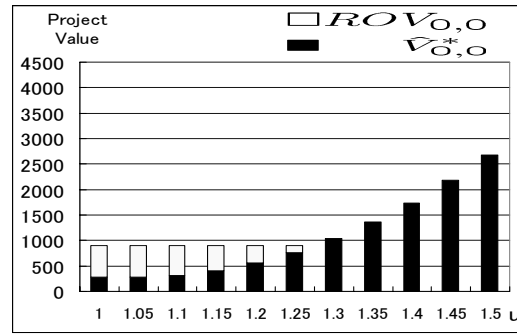

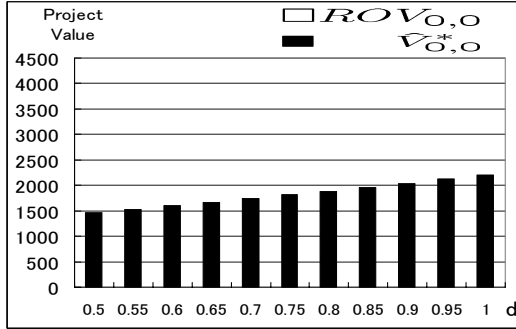Figure 7.4 Structural Properties about u of $\hat{V}_{0,0}^*$, $V_{0,0}^*$, and $ROV_{0,0}^*$

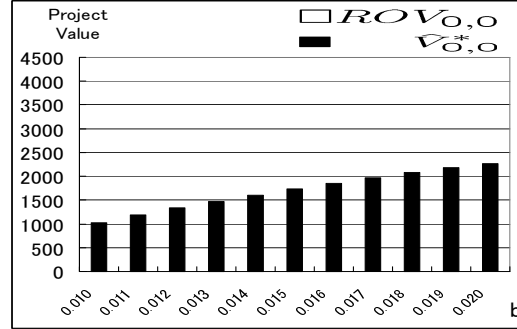Figure 7.5 Structural Properties about d of $\hat{V}_{0,0}^*$, $V_{0,0}^*$, and $ROV_{0,0}^*$

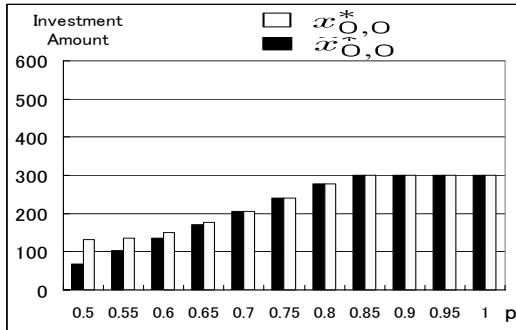Figure 7.6 Structural Properties about b of $\hat{V}_{0,0}^*$, $V_{0,0}^*$, and $ROV_{0,0}^*$

Figure 7.7 Structural Properties about p of $\hat{x}_{0,0}^*$ and $x_{0,0}^*$
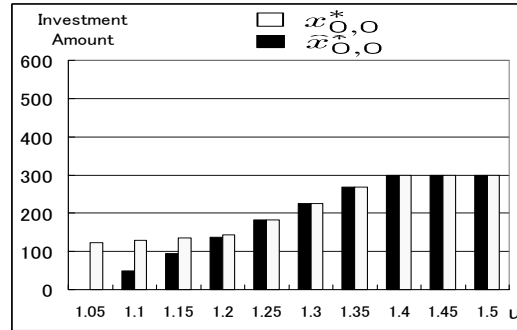

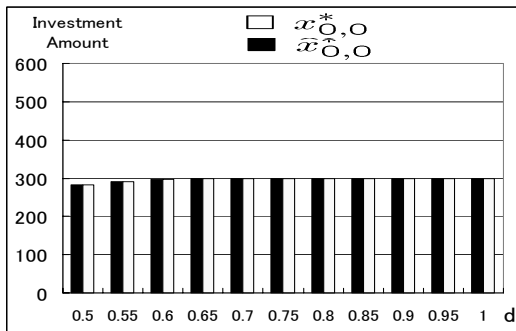Figure 7.8 Structural Properties about u of $\hat{x}_{0,0}^*$ and $x_{0,0}^*$

Figure 7.9 Structural Properties about d of $\hat{x}_{0,0}^*$ and $x_{0,0}^*$

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Appendix

In this appendix, we establish various lemmas which will provide useful tools for proving key theorems in the main text. Let

$$f(x, A_i, B_i) = G(\beta(x), A_i, B_i) - x \tag{A.1}$$
$$= \beta(x)A_i + \{1 - \beta(x)\}B_i - x, \quad i = 1, 2.$$

For notational convenience, we define

$$f_i(x) = (x, A_i, B_i) \ for \ all \ x \geq 0 \tag{A.2}$$

and

$$f_i(z_i^*) = \max_{x \geq 0} f_i(x); \ f_i(x_i^*) = \max_{0 \leq x \leq c} f_i(x), \quad i = 1, 2. \tag{A.3}$$

We recall that $\beta(x)$ is the probability that the project can be continued successfully for one period given that the investment amount for the period is x. It is assumed that, for $x \geq 0$,

$$\beta(0) = 0, \quad 0 \leq \beta(x) \leq 1, \quad \beta'(x) > 0, \quad \beta''(x) < 0, \tag{A.4}$$

where $\beta'(x) = \frac{d}{dx}\beta(x)$ and $\beta''(x) = \frac{d^2}{dx^2}\beta(x)$. It can be readily seen that f_i is strictly concave and has the unique maximum at z_i^* satisfying

$$f_i'(z_i^*) = \beta'(z_i^*)(A_i - B_i) - 1 = 0 \iff \beta'(z_i^*) = \frac{1}{(A_i - B_i)}. \tag{A.5}$$

As in the main text, we assume that

$$A_i > B_i, \quad i = 1, 2. \tag{A.6}$$

The following lemmas then hold true. Proofs are straightforword and omitted here.

Lemma A.1 *For $i = 1, 2$, let $f_i(x)$ be as in (A.1) and define z_i^* and x_i^* as in (A.2) and (A.3) respectively. One then has the followings.*

a) $x_i^* = \min\{z_i^*, c\}$.

b) x_i^* *is nondecreasing in* c

Lemma A.2 *For $i = 1, 2$, let $f_i(x)$ be as in (A.1). If $A_1 \geq A_2$ and $B_1 \geq B_2$, then $f_1(x_1^*) \geq f_2(x_2^*)$. Equality holds if and only if $A_1 = A_2$ and $B_1 = B_2$.*

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Lemma A.3 *Let $A_1 \geq A_2$. If either $B_1 = B_2$ or $B_1 > B_2$ and $A_1 - A_2 \geq B_1 - B_2$ holds true, then $x_1^* \geq x_2^*$.*

Lemma A.4 *Let $A_1 > A_2 > A_3$ and $B_1 > B_2 > B_3$. If*

$$A_1 - B_1 > A_2 - B_2 > A_3 - B_3, \quad \frac{A_1 + A_3}{2} > A_2 \ and \ \frac{B_1 + B_3}{2} > B_2, \tag{A.7}$$

then

$$\frac{f_1(x_1^*) + f_3(x_3^*)}{2} > f_2(x_2^*). \tag{A.8}$$

In order to observe monotonicity properties concerning p, we rewrite $f(x, A_i, B_i)$ as

$$f_i(x, p_i, A_i(p_i), B_i) = G(\beta(x), A_i(p_i), B_i) - x, \quad i = 1, 2. \tag{A.9}$$

For notational convenience, we define

$$f_i(x, p_i) = f(x, p_i, A_i(p_i), B_i), \quad i = 1, 2, \tag{A.10}$$

and

$$f_i(z_i^*, p_i) = \max_{x \geq 0} f_i(x, p_i); \ \ f_i(x_i^*, p_i) = \max_{0 \leq x \leq c} f_i(x, p_i), \quad i = 1, 2. \tag{A.11}$$

In this Appendix, we assume

$$A_i(p) = G(p, C_i, D_i); \ \ C_i > D_i > B_i; \ \ i = 1, 2, \ 0 \leq p \leq 1. \tag{A.12}$$

In parallel with Lemmas A.2 and A.3, we have the following lemma.

Lemma A.5 *Let $C_1 \geq C_2$, $D_1 \geq D_2$ and $p_1 > p_2$.*

a) *If $B_1 \geq B_2$, then $f_1(x_1^*, p_1) \geq f_2(x_2^*, p_2)$.*

b) *If $B_1 = B_2$, then $x_1^* \geq x_2^*$.*

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