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Abstract This paper presents a discrete time optimal asset selling problem with a predetermined final deadline where the process of selling the asset may reach a revised deadline at any point in time prior to the predetermined deadline with some probability. In this problem we assume that an appearing buyer offers a price to the seller for the asset in question. The seller then decides whether or not to sell the asset by comparing the offered price to his optimal reservation price. Our main focus is on clarifying the properties of the seller's optimal reservation price. In particular, we are interested in investigating the relationship between the seller's optimal reservation price and the probability of the process reaching a revised deadline unexpectedly before the predetermined deadline.

Keywords: Dynamic programming; reservation price; uncertain deadline

1 Introduction

Dealings of assets between seller and buyer are an essential part of any economy. Over the years, asset dealing problems have been attracting considerable attention among researchers and practitioners in the economics and operations research communities. A common basic assumption in research on this problem is that there exists a certain date in the future, the predetermined deadline, by which the asset must be sold or bought. The existence of such a deadline in the asset dealing problems is a realistic requirement in many real-life situations since the selling process cannot continue forever for reasons such as limited usability life of the asset caused by possible physical decaying, the seller's limited financial resources, and so on. Although in many circumstances a deadline is fixed before the selling process begins, in reality this deadline may change. This is because it is possible that an unfavorable event, which will substantially reduce the asset's market value, would occur unexpectedly prior to the deadline. As a result, the buyers demanding the asset would stop appearing. The seller would have no choice but to terminate the selling process instead of continuing it up to the previously fixed deadline. Examples of such unfavorable events include:

1. The sudden appearance of a cheaper and better quality asset that would substitute for the seller's asset. If such an event materialized, buyers demanding the seller's asset would stop appearing. The seller would be resigned to terminate the asset selling process by salvaging the asset.
2. A drastic drop in the asset value caused by a lost of one of the asset's valuable components. Consider the case of the owner of an advertising agency who has determined to put his firm on sale by a deadline. Knowing the fact, his key advertisement designers, the firm's most valuable asset, may leave the firm having been headhunted by a rival firm. If such situation occurred, the firm would lose its market value substantially because it would be difficult for the owner to find capable designers to replace those who had left. In this case, the buyers who consider the firm an attractive buy would

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stop appearing. The owner would have to liquidate the firm by salvaging what was left.

Taking the possible occurrence of such an event into consideration, in this paper we propose a model of the asset selling problem with a *predetermined* deadline where the process of selling the asset may reach a revised deadline at any next point in time prior to the predetermined deadline with a probability. Let us refer to this problem as the asset selling problem with an *uncertain deadline*. In our model, we assume that an appearing buyer offers a price to the seller for the asset in question. The seller then decides whether or not to sell the asset by comparing the offered price to his optimal reservation price. Here, the optimal reservation price means the seller's minimum permissible selling price; he is willing to sell the asset if and only if the price offered by the buyer is greater than or equal to his minimum permissible selling price.

The model proposed in this paper is closest in spirit to that of Iida and Mori [3] who formulated the multiperiods optimization problem using the Markov decision process in which probability distributions for the planning horizon are given. In other words, the planning horizon of the process is a random variable. In [3] it is shown that an optimal stationary strategy for the process may not exist. Our model is more modest than that of [3] in the sense that we limit the optimization problem to the asset selling problem in which the probability distribution for the planning horizon is a geometric distribution. Our main focus is on clarifying the properties of the seller's optimal reservation price. In particular, we are interested in investigating the relationship between the seller's optimal reservation price and the probability of the process reaching a revised deadline unexpectedly before the predetermined deadline. Furthermore, our model is also related in some respect to the literature on the conventional optimal stopping problem with no recall [7] [9] [12] [13] [14], and differs from the conventional ones in the sense that the deadline is uncertain in our model.

With respect to the seller's optimal reservation price, intuitively, we might expect that the lower the probability of the selling process reaching the revised deadline at the next point in time may be, the higher his optimal reservation price may become. A low probability leads to a longer selling period. This is beneficial for the seller because even if he raises his optimal reservation price, he still has ample time to find buyers who may make an offer that he would be willing to accept. Therefore, in this case, the profit earned from selling the asset may become greater. However, our analysis shows that this conjecture does not always hold. Indeed, one of the distinctive results obtained in this paper is that there exists a certain condition under which the higher the probability may be, the higher the optimal reservation price may become (see Lemma 4.4(d)).

The rest of this paper is organized as follows. Section 2 that follows provides a strict definition of the model examined in the paper. Section 3 defines several functions and examines their properties that will be used in the subsequent analysis. Section 4 prescribes and examines the optimal decision rule of our model. In Section 5 we provide some numerical examples that exemplify the properties of the optimal decision rule. Finally, in Section 6 we summarize the conclusions of our research and suggest some further work which could be done.

2 Model

Consider the following discrete-time sequential stochastic decision process where points in time are numbered backward from the final point in time of the planning horizon, time 0 (the deadline) as $0, 1, \cdots$

and so on. Accordingly, if time t is a present point in time, the two adjacent times $t+1$ and $t-1$ are the previous and next points in time, respectively. Let the time interval between times t and $t-1$ be called the period t. This is small enough that no more than one buyer may appear. A seller must sell his asset up to the deadline; for convenience, let it be called the *predetermined* deadline. While the deadline is defined as above, the asset selling process is assumed also to reach a revised deadline at the next point in time, prior to the predetermined deadline, with a probability q $(0 < q < 1)$, provided that the process has proceeded up to the present point in time. Let this revised deadline be called the *undetermined* deadline.

In this paper we assume that a buyer appears at each point in time with a probability λ $(0 < \lambda < 1)$. A buyer is also assumed to appear at the deadline, whether the predetermined or undetermined deadline, with the probability λ. Let w denote the price offered by an appearing buyer. Purchasing prices proposed by buyers appearing at successive points in time, $w,\ w', \cdots$, are independent identically distributed random variables having a known continuous distribution function $F(w)$ with a finite expectation μ; let $f(w)$ denote its probability density function, which is truncated on both sides. More precisely, $F(w)$ and $f(w)$ are defined as follows. For certain given numbers a and b such that $0 < a < b < \infty$

$$\begin{array}{llllll} F(w) = 0, & w \le a, & 0 < F(w) < 1, & a < w < b, & F(w) = 1, & b \le w, \\ f(w) = 0, & w < a, & f(w) > 0, & a \le w \le b, & f(w) = 0, & b < w. \end{array}$$

Then clearly $0 < a < \mu < b$.

In addition, the asset remaining unsold at time 0 can be sold at the salvage price ρ $(-\infty < \rho < \infty)$ where $\rho < 0$ implies the disposal cost to discard the unsold asset. Let $h \ge 0$ denote the inventory holding cost of the asset remaining unsold for a period. Moreover, by β $(0 < \beta \le 1)$ let us denote the discount factor, implying that the monetary value of one unit a period hence is equivalent to that of β units at the present point in time.

The objective here is to find the optimal decision rule to maximize the total expected present discounted net profit over the planning horizon, i.e., the total expected present discounted revenue *minus* the total expected present discounted holding cost.

3 Preliminaries

This section defines some functions used to describe the system of optimal equations of our model and clarifies their properties. These will be applied to the analysis of our model. First, for any real number x let us define:

$$T(x) = \mathbf{E}[\max\{w - x, 0\}], \tag{3.1}$$

$$K(x) = \lambda\beta T(x) - (1-\beta)x, \tag{3.2}$$

$$D(x) = K(x) + q(K(\rho) + \rho - K(x) - x) - h \tag{3.3}$$

where $\mathbf{E}[\ ]$ represents the taking of expectation with respect to w. Here note that

$$D(\rho) = K(\rho) - h. \tag{3.4}$$

Furthermore, by $x_K(h)$ and x_D, let us denote the solutions of the equations $K(x) = h$ and $D(x) = 0$, respectively, if they exist, i.e.,

$$K(x_K(h)) = h, \qquad D(x_D) = 0. \tag{3.5}$$

If $K(x) = h$ has multiple solutions, let us newly define the *minimum* of them by $x_K(h)$. The functions $T(x)$ and $K(x)$ are also defined in [7], which studied the asset selling problem with a fixed deadline. For the purpose of the analysis of our model, we find it useful to introduce some properties of $K(x)$ whose proofs can be found in [7].

Lemma 3.1 (Ikuta [7])

(a) *$K(x)$ is continuous and nonincreasing on $(-\infty, \infty)$.*

(b) *$K(x) + x$ is strictly increasing on $(-\infty, \infty)$.*

(c) *$|K(x) + x - K(y) - y| \leq \beta|x - y|$ for any x and y.*

(d) *If $(1-\beta)^2 + h^2 = 0$, then $x_K(h) = b$ where $x < (\geq)\ x_K(h) \Leftrightarrow K(x) > (=)\ h$.*

(e) *If $(1-\beta)^2 + h^2 \neq 0$, then $x_K(h)$ uniquely exists with $x_K(h) < b$ where $x < (= (>))\ x_K(h) \Leftrightarrow K(x) > (= (<))\ h$.*

Now, we examine the properties of the function $D(x)$ defined above which will be used in the analysis of the model.

Lemma 3.2

(a) *$D(x)$ is strictly decreasing on $(-\infty, \infty)$ and diverges to $-\infty$ (∞) as $x \to \infty\,(-\infty)$.*

(b) *x_D uniquely exists.*

(c) *$\rho < (= (>))\ x_K(h) \Leftrightarrow x_K(h) > (= (<))\ x_D$.*

(d) *Let $(1-\beta)^2 + h^2 = 0$.*

1. *$\rho < (\geq)\ x_K(h) \Rightarrow \rho < (=)\ x_D$.*
2. *$\rho < (\geq)\ x_K(h) \Rightarrow \rho < x_D < x_K(h) = b$ $(\rho = x_D \geq x_K(h) = b)$.*

(e) *Let $(1-\beta)^2 + h^2 \neq 0$.*

1. *$\rho < (= (>))\ x_K(h) \Rightarrow \rho < (= (>))\ x_D$.*
2. *$\rho < (= (>))\ x_K(h) \Rightarrow \rho < x_D < x_K(h) < b$ $(\rho = x_D = x_K(h) < b$ $(\rho > x_D > x_K(h)))$.*

Proof. (a) Since Eq. (3.3) can be rewritten as $D(x) = (1-q)K(x) - qx + q(K(\rho)+\rho) - h$, the assertion is immediate from the assumption of $1 > q > 0$ and Lemma 3.1(a).

(b) Evident from (a).

(c) Let $\rho < (= (>))\ x_K(h)$. Then from Lemma 3.1(b) and Eq. (3.5) we obtain $K(\rho)+\rho < (= (>))\ K(x_K(h))+ x_K(h) = h + x_K(h)$ or equivalently $K(\rho) + \rho - h - x_K(h) < (= (>))\ 0$. Further, from Eqs. (3.3) and (3.5) we get

$$\begin{aligned} D(x_K(h)) &= K(x_K(h)) + q(K(\rho) + \rho - K(x_K(h)) - x_K(h)) - h \\ &= q(K(\rho) + \rho - h - x_K(h)) < (= (>))\ 0 = D(x_D). \end{aligned}$$

Thus $x_K(h) > (= (<))\ x_D$ due to (a). The inverse holds by contraposition.

(d) Let $(1-\beta)^2 + h^2 = 0$, so $\beta = 1$ and $h = 0$. Then $x_K(h) = b$ from Lemma 3.1(d).

(d1) Suppose $\rho < (\geq)\ x_K(h)$. Then $K(\rho) > (=)\ h$ due to Lemma 3.1(d). Therefore, from Eq. (3.4) we get $D(\rho) > (=)\ 0 = D(x_D)$. Thus we have $\rho < (=)\ x_D$ due to (a, b).

(d2) Suppose $\rho < (\geq)\ x_K(h)$. Then since $\rho < (=)\ x_D$ from (d1) and $x_D < (\geq)\ x_K(h)$ from (c), we have $\rho < x_D < x_K(h) = b$ ($\rho = x_D \geq x_K(h) = b$).

(e) Let $(1-\beta)^2 + h^2 \neq 0$. Then $x_K(h) < b$ from Lemma 3.1(e).

(e1) Suppose $\rho < (= (>))\ x_K(h)$. Then $K(\rho) > (= (<))\ h$ due to Lemma 3.1(e). Hence from Eq. (3.4) we obtain $D(\rho) > (= (<))\ 0 = D(x_D)$. Thus $\rho < (= (>))\ x_D$ due to (a, b).

(e2) Suppose $\rho < (= (>))\ x_K(h)$. Then since $\rho < (= (>))\ x_D$ due to (e1) and $x_D < (= (>))\ x_K(h)$ due to (c), we have $\rho < x_D < x_K(h) < b$ ($\rho = x_D = x_K(h) < b$ ($\rho > x_D > x_K(h)$)). ∎

4 Analysis

In this section, we first discuss the finite planning horizon model and then the limiting planning horizon model.

4.1 Finite planning horizon model

Suppose the asset selling process starts from a time t ($1 \leq t < \infty$). When the current point in time has not yet become the deadline, let u_t and U_t be the maximum expected present discounted profits obtained over the planning horizon, starting, respectively, with no buyer and with a buyer. If the current point in time has become the deadline with a probability q, the process terminates after having dealt with a buyer who offers a price w appearing with a probability λ at that time; if no buyer appears at that time, the seller must compulsorily sell the asset to a salvage dealer at the price ρ. Then we have

$$u_0 = \rho, \tag{4.1}$$

$$u_t = \beta\Big((1-q)(\lambda U_{t-1} + (1-\lambda)u_{t-1}) + q\big(\lambda\,\mathbf{E}[\max\{w,\rho\}] + (1-\lambda)\rho\big)\Big) - h, \qquad t \geq 1, \tag{4.2}$$

$$U_t = \mathbf{E}[\max\{w, u_t\}], \qquad t \geq 0, \tag{4.3}$$

Noting $\max\{w, u_t\}$ of Eq. (4.3), we see that the optimal decision rule of the model for time $t \geq 0$ can be prescribed as follows.

Optimal Decision Rule *For an appearing buyer with an offer w, if $w \geq u_t$, sell the asset to the buyer, or else do not; in other words, u_t becomes the seller's* optimal reservation price.

Since $\mathbf{E}[\max\{w,\rho\}] = \mathbf{E}[\max\{w-\rho, 0\}] + \rho = T(\rho) + \rho$ and $\mathbf{E}[\max\{w, u_t\}] = \mathbf{E}[\max\{w - u_t, 0\}] + u_t = T(u_t) + u_t$ for $t \geq 0$, we can rewrite Eqs. (4.3) and (4.2) as, respectively,

$$U_t = T(u_t) + u_t, \qquad t \geq 0, \tag{4.4}$$

$$u_t = (1-q)\beta(\lambda T(u_{t-1}) + u_{t-1}) + q\beta(\lambda T(\rho) + \rho) - h, \qquad t \geq 1. \tag{4.5}$$

Since $K(x) + x = \beta(\lambda T(x) + x)$ from Eq. (3.2), we can rearrange Eq. (4.5) as follows.

$$\begin{aligned} u_t &= (1-q)\big(K(u_{t-1}) + u_{t-1}\big) + q(K(\rho) + \rho) - h & (4.6)\\ &= K(u_{t-1}) + u_{t-1} + q(K(\rho) + \rho - K(u_{t-1}) - u_{t-1}) - h & (4.7)\\ &= u_{t-1} + D(u_{t-1}), \qquad t \geq 1 \quad \text{(see Eq. (3.3))}. & (4.8) \end{aligned}$$

Now, let us proceed to examining the monotonicity of u_t in $t \geq 0$.

Lemma 4.1

(a) *Let $(1-\beta)^2 + h^2 = 0$. If $\rho < (\geq)\ x_K(h)$, then u_t is strictly increasing (constant) in $t \geq 0$.*

(b) *Let $(1-\beta)^2+h^2 \neq 0$. If $\rho < (= (>))\ x_K(h)$, then u_t is strictly increasing (constant (strictly decreasing)) in $t \geq 0$.*

Proof. Noting Eq. (4.1), from Eq. (4.6) we get

$$u_1 = (1-q)(K(\rho)+\rho) + q(K(\rho)+\rho) - h = K(\rho) + \rho - h. \tag{4.9}$$

(a) Let $(1-\beta)^2+h^2 = 0$. If $\rho < (\geq)\ x_K(h)$, then $K(\rho) > (=)\ h$ due to Lemma 3.1(d). Therefore, from Eqs. (4.9) and (4.1) we obtain $u_1 > (=)\ \rho = u_0$. Suppose $u_{t-1} > (=)\ u_{t-2}$. Then from Eq. (4.6) and Lemma 3.1(b) we get $u_t > (=)\ (1-q)\big(K(u_{t-2}) + u_{t-2}\big) + q(K(\rho)+\rho) - h = u_{t-1}$. Accordingly, the assertion holds by induction.

(b) Let $(1-\beta)^2+h^2 \neq 0$. If $\rho < (= (>))\ x_K(h)$, then $K(\rho) > (= (<))\ h$ due to Lemma 3.1(e). Hence from Eq. (4.9) we obtain $u_1 > (= (<))\ \rho = u_0$. Using the result, we can prove the assertion in almost the same way as in (a). ∎

Intuition suggests that in order to avoid having leftover item at the deadline, a seller may become more compelled to sell if the deadline is approaching, implying that he will lower his optimal reservation price u_t as the remaining time periods up to the deadline t decreases. Therefore, it can be conjectured that u_t is strictly increasing in t. The result obtained from our analysis turns out to be affirmative for the case of $\rho < x_K(h)$. This result is similar in some respects to the one in the literature on the asset selling problem with a fixed deadline [5] [6] [13] where u_t is proven to be nondecreasing in t. However, we explicitly derive the conditions on which u_t is strictly increasing and constant in t.

Furthermore, Lemma 4.1 shows that if $\rho \geq x_K(h)$, then the above stated conjecture does not hold; in this case u_t is strictly decreasing or constant in t. Here, we propose a reasoning for the occurrence of the counterintuitive phenomenon. If the salvage price is sufficiently high, the seller may raise his optimal reservation price as the deadline draws near since he is able to sell the item at the high salvage price by rejecting all appearing buyers' offers. This situation makes the seller more tempted to raise his optimal reservation price so that only the offer with a value greater than the high salvage price may be accepted in order to gain greater profit from selling the asset. However, it should be noted that when $\rho = x_K(h)$ or "$(1-\beta)^2+h^2 = 0$ and $\rho > x_K(h)$", the above reasoning does not hold however large the salvage price ρ may be; in this case his optimal reservation price u_t becomes independent of t.

Next, we investigate the monotonicity of u_t in q for $t \geq 0$.

Lemma 4.2

(a) *u_0 and u_1 are q-independent.*

(b) *u_t is independent of q for $t \geq 0$ if $\rho = x_K(h)$ or if $\rho > x_K(h)$ and $(1-\beta)^2+h^2 = 0$.*

(c) *Let $\rho < x_K(h)$. Then u_2 is strictly decreasing in q.*

(d) *Let $\rho > x_K(h)$ and $(1-\beta)^2+h^2 \neq 0$. Then u_2 is strictly increasing in q.*

Proof. Note that Eq. (4.7) with $t = 2$ can be expressed as $u_2 = K(u_1) + u_1 + q(K(\rho) + \rho - K(u_1) - u_1) - h \cdots (1^*)$.

(a) Evident from Eqs. (4.1) and (4.9).

(b) Immediate from the fact that $u_t = u_0 = \rho$ for $t \geq 0$ from Lemma 4.1 and Eq. (4.1).

(c) Let $\rho < x_K(h)$. Then $\rho = u_0 < u_1$ from Lemma 4.1(a,b) and Eq. (4.1), hence $K(\rho) + \rho < K(u_1) + u_1$ due to Lemma 3.1(b) or equivalently $K(\rho) + \rho - K(u_1) - u_1 < 0$. Since u_1 is q-independent due to (a), it follows that both $K(u_1) + u_1$ and $K(\rho) + \rho - K(u_1) - u_1$ are also q-independent. From this result and (1*) we immediately see that the assertion is true.

(d) Let $\rho > x_K(h)$ and $(1-\beta)^2 + h^2 \neq 0$. Then since $\rho = u_0 > u_1$ from Lemma 4.1(b) and Eq. (4.1), we have $K(\rho) + \rho - K(u_1) - u_1 > 0$ in the same way as in (c), hence the assertion holds. ∎

In Lemma 4.2 we showed that if $\rho = x_K(h)$ or if $\rho > x_K(h)$ and $(1-\beta)^2 + h^2 = 0$, then the seller's optimal reservation price u_t is independent of q for *all* $t \geq 0$. When these conditions are not satisfied, although it will be shown that u_t may be monotonic in q as $t \to \infty$ in the limiting planning horizon model; however, in the finite planning horizon model the monotonicity can be successfully verified only for $t = 0$, 1, and 2. Furthermore, in Section 5 we will show that u_t for $t \geq 3$ may possess a similar pattern of monotonicity with that of u.

4.2 Limiting planning horizon model

In this subsection, we clarify the properties of the optimal decision rule for the model with a limiting planning horizon by taking the limit of u_t as $t \to \infty$. In this model, an asset could be on sale for a sufficiently long time period unless an unfavorable event, which makes the seller decide to exit the market, occurs. Here let $u = \lim_{t \to \infty} u_t$, the seller's limiting optimal reservation price, if it exists. For the limiting planning horizon model we are interested in investigating the monotonicity of u in q. Firstly, we prove that a finite u exists.

Lemma 4.3 *u_t converges to a finite u as $t \to \infty$ where $u = x_D$.*

Proof. Note that $(1-q)\beta < 1$ due to the assumption of $q < 1$. For convenience, by $\mathcal{L}(x)$ let us denote the right-hand side of Eq. (4.6), i.e.,

$$\mathcal{L}(x) \;=\; (1-q)(K(x) + x) + q(K(\rho) + \rho) - h.$$

Then, noting Lemma 3.1(c), we get

$$|\mathcal{L}(x) - \mathcal{L}(y)| \;=\; (1-q)|K(x) + x - K(y) - y| \leq (1-q)\beta|x - y|.$$

Hence it follows that $\mathcal{L}$ is a contraction mapping. Accordingly, the former half of the assertion holds where u is given by the unique solution of the equation

$$u \;=\; (1-q)(K(u) + u) + q(K(\rho) + \rho) - h.$$

The above equation can be rewritten as

$$u \;=\; K(u) + u + q(K(\rho) + \rho - K(u) - u) - h,$$

from which we obtain $D(u) = 0$ (see Eq. (3.3)). Therefore, $u = x_D$ due to Eq. (3.5) and Lemma 3.2(b). ∎

Next, in order to examine the relationship of the limiting optimal reservation price u with q, we clarify the properties of x_D and $D(\rho)$.

Lemma 4.4

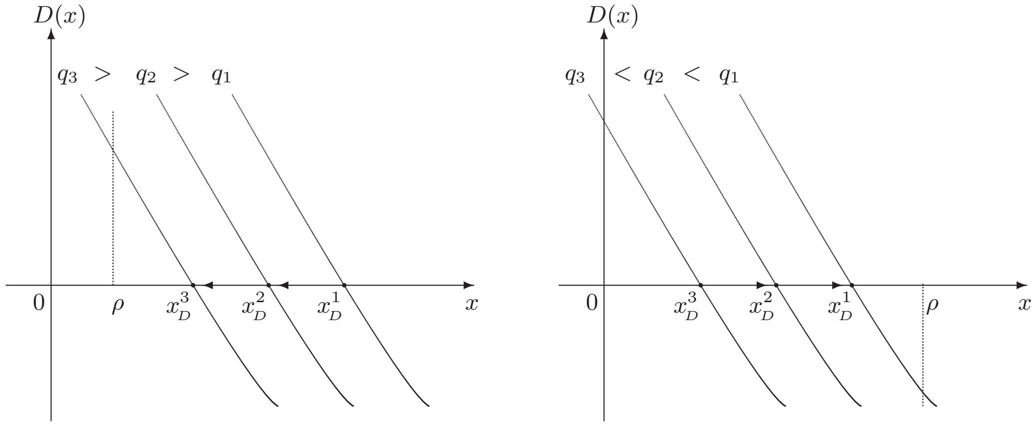
(a) *If $D(\rho) > (<)\ 0$ for all q, then x_D is strictly decreasing (strictly increasing) in q with $x_D > (<)\ \rho$.*

(b) *If $D(\rho) = 0$ for all q, then $x_D\ (= \rho)$ is independent of q.*

(c) *Let $(1-\beta)^2 + h^2 = 0$. If $\rho < (\geq)\ x_K(h)$, then u is strictly decreasing in (independent of) q.*

(d) *Let* $(1-\beta)^2+h^2 \neq 0$*. If* $\rho < (= (>))\ x_K(h)$*, then* u *is strictly decreasing in* (*independent of* (*strictly increasing in*)) q*.*

Proof. (a) If $D(\rho) > (<)\ 0$ for all q, then $x_D > (<)\ \rho$ for all q from Lemma 3.2(a,b) and the definition of x_D. Let $x > (<)\ \rho$. Then $K(x) + x > (<)\ K(\rho) + \rho$ from Lemma 3.1(b), i.e., $K(\rho) + \rho - K(x) - x < (>)\ 0$, implying that $D(x)$ is strictly decreasing (strictly increasing) in q for $x > (<)\ \rho$ (see Eq. (3.3)). From this result and Lemma 3.2(a), it follows that x_D is strictly decreasing (strictly increasing) in q for $x > (<)\ \rho$ (see Figure 4.1).



I: Let $D(\rho) > 0$ for all q. II: Let $D(\rho) < 0$ for all q.

Figure 4.1: Graphs of $D(x)$ in x for $0 < q < 1$.

(b) If $D(\rho) = 0$ for all q, then $x_D = \rho$ for all q from Lemma 3.2(b) and Eq. (3.5), hence the assertion holds.

(c) Let $(1-\beta)^2 + h^2 = 0$ and $\rho < (\geq)\ x_K(h)$. Then $K(\rho) > (=)\ h$ due to Lemma 3.1(d), hence from Eq. (3.4) we get $D(\rho) > (=)\ 0$ for all q. Accordingly the assertion holds from (a,b) and the fact that $u = x_D$ from Lemma 4.3.

(d) Let $(1-\beta)^2 + h^2 \neq 0$ and $\rho < (= (>))\ x_K(h)$. Then $K(\rho) > (= (<))\ h$ due to Lemma 3.1(e), hence from Eq. (3.4) we get $D(\rho) > (= (<))\ 0$ for all q. Accordingly the assertion holds from (a,b) and the fact that $u = x_D$ from Lemma 4.3. ∎

With reference to Lemma 4.4(c,d), we can observe the followings:

1. Let $(1-\beta)^2 + h^2 = 0$. If $\rho < x_K(h)$, then the seller's limiting optimal reservation price u is strictly decreasing in q (Lemma 4.4(c)), implying that he is more willing to lower his limiting optimal reservation price as q increases. As a practical matter, this suggests that as q increases, the offer proposed by an appearing buyer is more likely to be accepted by the seller. On the other hand, if $\rho \geq x_K(h)$, since u is independent of q, changes in q does not have effect on u.
2. Let $(1-\beta)^2 + h^2 \neq 0$. If $\rho < (>)\ x_K(h)$, then u is strictly decreasing (strictly increasing) in q (Lemma 4.4(d)), implying that the seller lowers (raises) his limiting optimal reservation price as q increases. This indicates that as q increases, the seller is more likely to accept (reject) an appearing buyer's offer. However, if $\rho = x_K(h)$, since u is independent of q, changes in q does not have effect on u.

Intuition suggests that the higher the probability that the next point in time becomes the undetermined deadline, the seller, if he has an asset on hand, would be willing to lower his limiting optimal reservation price in order to avoid having leftovers at the undetermined deadline. This is because the higher the probability is, the lesser is the time available for the seller to be met by the buyer who would offer a price which is acceptable by him. Therefore, it can be conjectured that the higher the probability q is, the lower the limiting optimal reservation price will be, that is the limiting optimal reservation price is strictly decreasing in q. However, through our analysis, it is shown that the above conjecture does not always hold.

From Lemma 4.4(c,d), we should note that our result is consistent with the conjecture stated above when $\rho < x_{K}(h)$. However, the conjecture fails to hold if $\rho \geq x_{K}(h)$ where u is either independent of q or strictly increasing in q. Here, we propose a reasoning for the occurrence of the case ($\rho > x_{K}(h)$ and $(1-\beta)^2+h^2 \neq 0$) where u is strictly increasing in q. If the salvage price is sufficiently large, the seller may be more willing to hold the asset and only sell it to the salvage dealer when the undetermined deadline materializes. Accordingly, the seller may raise his limiting optimal reservation price as the probability q increases since he is able to sell the item at the high salvage price by rejecting all appearing buyers' offers.

In addition, we obtain a similar observation to the one related to the monotonicity of the seller's optimal reservation price u_t in q for the finite planning horizon model when $\rho = x_{K}(h)$ or '$(1-\beta)^2+h^2 = 0$ and $\rho > x_{K}(h)$'. Under these conditions, the above stated reasoning does not hold for the monotonicity of the seller's limiting optimal reservation price in q however large the salvage price ρ may be. His limiting optimal reservation price becomes independent of q due to Lemma 4.4(c,d). This implies that the seller will always hold the same limiting optimal reservation price irrespective of whether or not the deadline will materialize in near future.

5 Numerical Example

In this section, through numerical experiments let us exemplify the properties of the optimal decision rules for the finite and limiting planning horizon models. Let $\beta = 0.99$, $\lambda = 0.4$, $h = 0.4$, and let $F(w)$ be the uniform distribution on $[1.5, 2.5]$, i.e., $a = 1.5$ and $b = 2.5$. Then we have $(1-\beta)^2 + h^2 \neq 0$. Using Eq. (3.2), we can easily obtain $x_{K}(h) \approx 0.9655$.

1. *Monotonicity of u_t in t*: Let $q = 0.3$. Figure 5.2 depicts the monotonicity of u_t in t for various values of ρ. From this figure we see that in the finite planning horizon model the optimal reservation price u_t is constant for $t \geq 0$ if $\rho = x_{K}(h)$, while u_t is strictly increasing (strictly decreasing) in t if $\rho <$ ($>$) $x_{K}(h)$ (see Lemma 4.1(b)).

2. *Monotonicity of u_t and u in q*: Figures 5.3 and 5.4 depict the monotonicity of u and u_3 in $q \in [0.01, 0.99]$ for various values of ρ. From Figure 5.3 we see that in the limiting planning horizon model the seller's limiting optimal reservation prices u are independent of q if $\rho = x_{K}(h)$, while u is strictly decreasing in (strictly increasing in) q if $\rho <$ ($>$) $x_{K}(h)$ (see Lemma 4.4(d)). Although the monotonicity of the seller's optimal reservation price in q for $t = 3$ for the finite planning horizon model cannot be proven as stated earlier, from Figure 5.4 we see that u_3 demonstrates a similar pattern of monotonicity with that of u.

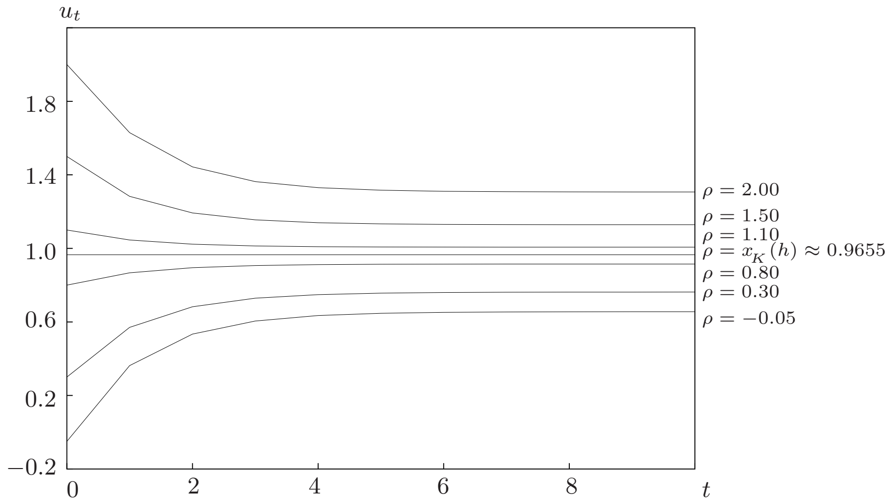


Figure 5.2: Monotonicity of u_t in t

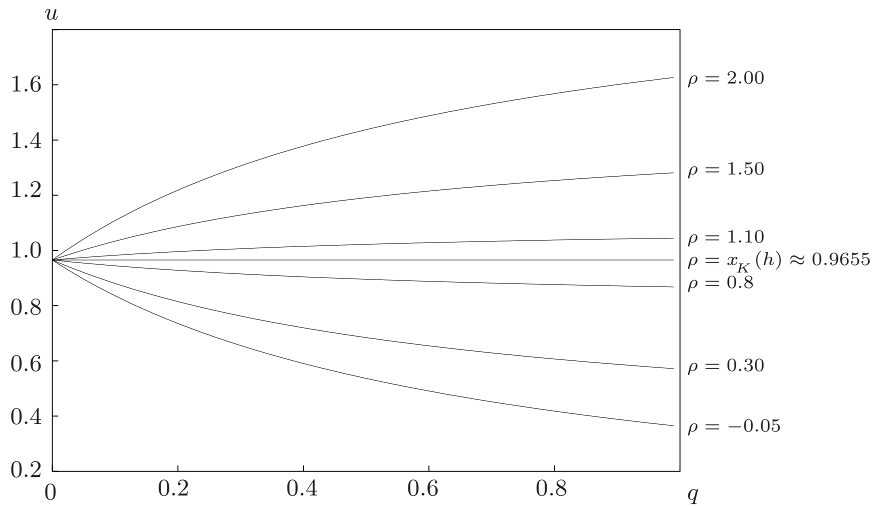


Figure 5.3: Monotonicity of u in q

6 Conclusions and Suggested Future Studies

In this paper we have proposed two models of the asset selling problem with an uncertain deadline: a finite planning horizon model and limiting planning horizon model. In Table 6.1 we shall summarize the conditions for the monotonicity of the optimal reservation price u_t in $t \geq 0$ and limiting optimal reservation price u in q.

In addition, if $\rho = x_K(h)$ or if $\rho > x_K(h)$ and $(1-\beta)^2 + h^2 = 0$, then the seller's optimal reservation price u_t is independent of q for *all* $t \geq 0$. When these conditions are not satisfied, in the limiting planning horizon model the u_t may be monotonic in q as $t \to \infty$; however, in the finite planning horizon model the monotonicity can be successfully verified only for $t = 0$, 1, and 2.

Now, we conclude with a discussion of some directions in which our model could be extended to make it more practical. Firstly, a model into which a cost of attracting buyers is incorporated, called the search cost, may be worth investigating. The introduction of a search cost inevitably would yield

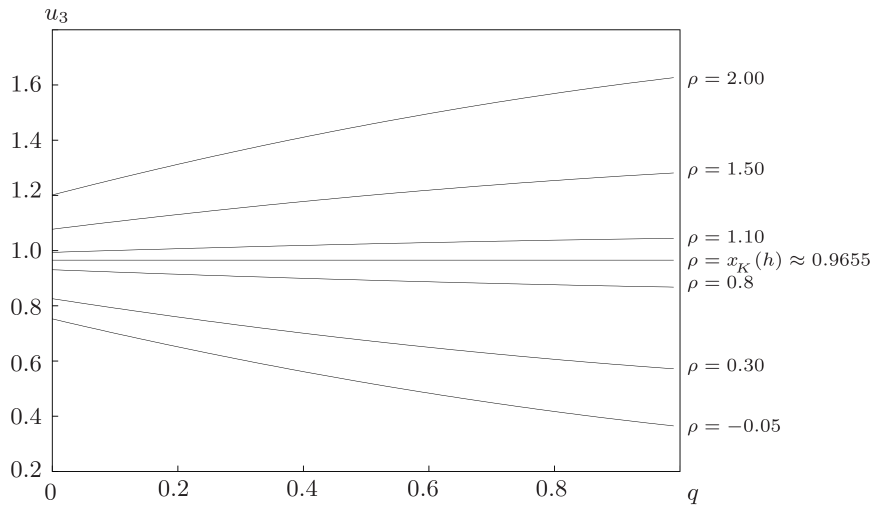


Figure 5.4: Monotonicity of u_3 in q

Table 6.1: Conditions for the monotonicity of u_t in $t \geq 0$ and u in q.

Condition	Finite planning horizon model	Limiting planning horizon model
$\rho < x_K(h)$	u_t is strictly increasing in $t \geq 0$.	u is strictly decreasing in q.
$\rho = x_K(h)$ $\rho > x_K(h)$ and $(1-\beta)^2 + h^2 = 0$	u_t is constant in $t \geq 0$.	u is independent of q.
$\rho > x_K(h)$ and $(1-\beta)^2 + h^2 \neq 0$	u_t is strictly decreasing in $t \geq 0$.	u is strictly increasing in q.

the option whether to conduct or to skip the search. Some models have been proposed for the optimal stopping problem with a fixed deadline and search cost where the search cost is fixed [1] [2] [7] [13] or total search budget is optimally allocated [5]. A model into which the assumption in [5] (the allocation of limited search budget) is introduced is also worth discussing. In addition, another possible extension would be to consider the future availability of the buyer who leaves the selling process without purchase but may return; the articles on the optimal stopping problem with recall and a certain deadline include [4], [8], [10], and [11].

References

[1] M. A. Arnold and S. A. Lippman. Selecting a selling institution: auctions versus sequential search. *Economic Inquiry*, 33:1–23, 1995.

[2] M. A. Arnold and S. A. Lippman. The analytics of search with posted prices. *Economic Theory*, 17:447–466, 2001.

[3] T. Iida and M. Mori. Markov decision processes with random horizon. *Journal of Operations Research Society of Japan*, 39(4):592–603, 1996.

[4] S. Ikuta. Optimal stopping problem with uncertain recall. *Journal of the Operations Research Society of Japan*, 31(2):145–170, 1988.

[5] S. Ikuta. The optimal stopping problem in which the sum of the accepted offer's value and the remaining search budget is an objective function. *Journal of the Operations Research Society of Japan*, 35(2):172–193, 1992.

[6] S. Ikuta. Optimal stopping problem with several search areas. *Journal of the Operations Research Society of Japan*, 38(1):89–106, 1995.

[7] S. Ikuta. An integration of the optimal stopping problem and the optimal pricing problem. *Discussion paper, No.1084, University of Tsukuba, Institute of Policy and Planning Sciences*, 2004.

[8] B. K. Kang. Optimal stopping problem with recall cost. *European Journal of Operations Research*, 117(2):222–238, 1999.

[9] S. Karlin. Stochastic models and optimal policy for selling an asset. In: *Studies in applied probability and management science* (J.A. Arrow, S. Karlin and H. Scarf, eds) 1962.

[10] E. Karni and A. Schwartz. Search theory: the case of search with uncertain recall. *Journal of Economic Theory*, 16:38–52, 1977.

[11] M. Landsberger and D. Peled. Duration of offers, price structure, and the gain from search. *Journal of Economic Theory*, 16:17–37, 1977.

[12] B. Leonard. To stop or not to stop, 1973.

[13] S. A. Lippman and J. J. McCall. The economics of job search: a survey. *Economic Inquiry*, 14:155–189, 1976.

[14] M. Sakaguchi. Dynamic programming of some sequential sampling design. *Journal of Mathematical Analysis and Applications*, 2:446–466, 1961.