平面分割の数え上げ問題と行列式・パフィアン

Masao Ishikawa[†]

[†]Department of Mathematics
Tottori University

combin.jp SUMMER SCHOOL 2008,
Culture Resort Festone, Okinawa, Japan.

Abstract

平面分割 (plane partitions) の数え上げ問題は MacMahon が研究を始めて以来、古典的な離散数学の問題として研究されてきたが、対称関数・群の表現論・数理物理などの分野にも現れる組合せ論的側面の研究対象でもある。この話の中では、MacMahon に始まる平面分割の母関数の古典論から始めていろいろな対称性を考慮した平面分割の母関数を、対称関数の応用して得る方法について述べ、その表現論や組合せ論との関係を振り返る。さらに、それらの応用として Mills-Robbins-Rumsey によって提出された totally symmetric self-complementary plane partitions や cyclically symmetric transpose-complementary plane partitions など交代符号行列 (alternating sign matrix) との関連を予想される平面分割の数え上げ問題を扱うことを目標にする。それらの母関数として、行列式・パフィアンによる表示や constant term による表示が得られるが、それらの行列式・パフィアンの計算は Plucker 関係式や discrete Hirota equation などの可積分系との深い関連が予想される。また、最近では affine Hecke algebra などの代数的側面との関係も数理物理学者達によって研究されている。

Plan of My Talk

1. Symmetric functions and
2. Plane partitions and symmetries
3. Totally symmetric self-complementary plane partitions

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Definition

The q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n = \prod_{i=1}^{n}(1-aq^{i-1})$$

and the q-binomial coefficeints is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_r(q;q)_{n-r}}.$$

Proposition

Let $\mathbb{A} = \{a_1, \ldots, a_n\}$ be an alphabet. If we put $a_i = q^{i-1}$ $(i = 1, \ldots, n)$, then we have

$$S^r(\mathbb{A}) = \begin{bmatrix} n+r-1 \\ r \end{bmatrix}_q$$

$$\Lambda^r(\mathbb{A}) = q^{\frac{r(r-1)}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q$$

Plane Partitions in $B(r,s,t)$

Assume a plane partition π is in $B(r,s,t)$. For example, the following π is in $B(4,5,3)$:

3	3	2	2	1
3	2	2	1	1
2	1	1	1	0
2	1	1	0	0

Add r to each entry in the first row, add $r-1$ to each entry in the second row, and so on, then we obtain the following column-strict plane partition π':

7	7	6	6	5
6	5	5	4	4
4	3	3	3	2
3	2	2	1	1

Plane Partitions in $B(r, s, t)$

Replace each entry π'_{ij} with $r + t + 1 - \pi'_{ij}$. Then we obtain a tableau T of shape r^s with each parts $\leq t$.

1	1	2	2	3
2	3	3	4	4
4	5	5	5	6
5	6	6	7	7

This gives a bijection $\pi \to T$ from the set of plane partitions in $B(r, s, t)$ to the set of tableaux of shape r^s with each parts $\leq t$. Thus the generating function

$$\mathrm{GF}(r, s, t) = \sum_{\pi \subseteq B(r,s,t)} q^{|\pi|}$$

is given by $q^{-\frac{r(r-1)s}{2}} S_{s^r}\left(1, q, q^2, \dots, q^{r+t-1}\right)$.

If we use

$$S_{\lambda/\mu}(\mathbb{A}) = \det\left[S^{\lambda_i-\mu_j-i+j}(\mathbb{A})\right]_{1\le i,j\le n},$$

then we obtain

$$\begin{aligned} \mathrm{GF}(r,s,t) &= q^{-\frac{r(r-1)s}{2}} S_{s^r}\left(1, q, q^2, \dots, q^{r+t-1}\right) \\ &= q^{-\frac{r(r-1)s}{2}} \det\left[S^{s-i+j}\left(1, q, q^2, \dots, q^{r+t-1}\right)\right]_{1\le i,j\le r} \\ &= q^{-\frac{r(r-1)s}{2}} \det\left(\begin{bmatrix} r+s+t-i+j-1 \\ s-i+j \end{bmatrix}_q\right)_{1\le i,j\le r} \end{aligned}$$

q-hook formula

Meanwhile, the q-hook formula tells us

$$s_\lambda\left(1,q,q^2,\dots,q^{n-1}\right)=q^{n(\lambda)}\prod_{x\in\lambda}\frac{1-q^{n+c(x)}}{1-q^{h(x)}}$$

where $n(\lambda)=\sum_{i\geq 1}(i-1)\lambda_i$. This shows

$$\mathrm{GF}(r,s,t)=\prod_{x\in(s^r)}\frac{1-q^{n+c(x)}}{1-q^{h(x)}},$$

which immediately implies

$$\mathrm{GF}(r,s,t)=\prod_{p\in B(r,s,t)}\frac{1-q^{\mathrm{ht}(p)+1}}{1-q^{\mathrm{ht}(p)}},$$

where $\mathrm{ht}(p)=\mathrm{ht}(x,y,z)=x+y+z-2$.

q-hook formula

Hook length and contents

8	7	6	5	4
7	6	5	4	3
6	5	4	3	2
5	4	3	2	1

0	1	2	3	4
−1	0	1	2	3
−2	−1	0	1	2
−3	−2	−1	0	1

Problem

Directly evaluate the above determinant to be the above product.

Definition

Let $n = 2r$ be an even integer and let $A = (a_{ij})_{1 \le i,j \le n}$ be an n by n skew symmetric matrix (i.e. $a_{ji} = -a_{ij}$), whose entries a_{ij} are in a commutative ring. The *Pfaffian* $\mathrm{Pf}(A)$ of A is defined by

$$\mathrm{Pf}(A) = \sum \epsilon(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)\, a_{\sigma_1\sigma_2} \dots a_{\sigma_{n-1}\sigma_n},$$

where the summation is over all partitions $\{\{\sigma_1, \sigma_2\}_<, \dots, \{\sigma_{n-1}, \sigma_n\}_<\}$ of $[n]$ into 2-element blocks, and $\epsilon(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)$ denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} & \sigma_n \end{pmatrix}.$$

Pfaffians

Example

For instance, when $n = 4$, the equation above reads:

$$\operatorname{Pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{14} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Definition

A permutation $(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)$ which arises from a partition of $[n]$ into 2-element blocks is called a *perfect matching* or a *1-factor*. We say that the points σ_{2i-1} and σ_{2i} are *connected* to each other in this perfect matching σ. We can express a perfect matching graphically by arranging the lattice points $1, \dots, n$ along the x-axis in the plane and representing the edges $(\sigma_{2i-1}, \sigma_{2i})$ by curves in the upper half plane. Two edges $(\sigma_{2i-1}, \sigma_{2i})$ and $(\sigma_{2j-1}, \sigma_{2j})$ in σ will be said to be *crossed* if the corresponding edges intersect in such an embedding. It is known that $\operatorname{sgn}\sigma$ agrees with $(-1)^k$ where k denotes the number of crossed pairs of edges in σ. We write $\mathscr{F}_n$ for the set of perfect matchings of $[n]$.

Example

For an example, the graphical representation of the perfect matching $\sigma = \{(1,4),(2,5),(3,6)\} \in \mathscr{F}_6$ is Figure 1 bellow, and its sign is -1.

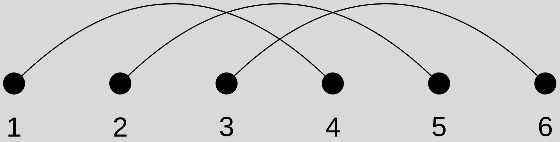

Figure: A perfect matching

Expansion formula

Definition

Let n be an even integer, and let A be a skew symmetric matrix of size n. For $1 \le i < j \le n$, let $(A; \{i,j\}, \{i,j\})$ denote the $(n-2)$ by $(n-2)$ skew symmetric sub-matrix obtained by removing both the ith and jth rows and both the ith and jth columns of A, i.e. $(A; \{i,j\}, \{i,j\}) = A^{\overline{\{i,j\}}}_{\overline{\{i,j\}}}$.
Let us define $\gamma(i,j)$ by

$$\gamma(i,j) = (-1)^{i+j-1} \mathrm{Pf}\,(A; \{i,j\}, \{i,j\})$$

for $1 \le i < j \le n$. We define the values of $\gamma(i,j)$ for $1 \le j \le i \le n$ so that $\gamma(j,i) = -\gamma(i,j)$ always holds.

Expansion formula

Proposition

Let n be an even integer and $A = (a_{ij})$ be an n by n skew symmetric matrix. For any i, j we have

$$\delta_{ij}\mathrm{Pf}(A) = \sum_{k=1}^{n} a_{kj}\gamma(k,i),$$

$$\delta_{ij}\mathrm{Pf}(A) = \sum_{k=1}^{n} a_{ik}\gamma(j,k).$$

Example

$$\mathrm{Pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ & & 0 & a_{34} & a_{35} & a_{36} \\ & & & 0 & a_{45} & a_{46} \\ & & & & 0 & a_{56} \\ & & & & & 0 \end{pmatrix}$$

$$= a_{12}\mathrm{Pf}\begin{pmatrix} 0 & a_{34} & a_{35} & a_{36} \\ & 0 & a_{45} & a_{46} \\ & & 0 & a_{56} \\ & & & 0 \end{pmatrix} - a_{13}\mathrm{Pf}\begin{pmatrix} 0 & a_{24} & a_{25} & a_{26} \\ & 0 & a_{45} & a_{46} \\ & & 0 & a_{56} \\ & & & 0 \end{pmatrix} + \cdots$$

Fundamental relation between Pfaffians and determinants

Proposition

For $m \times n$ matrix B ($m + n$ is even), we have

$$\mathrm{Pf}\begin{pmatrix} O_m & B \\ -{}^t B & O_n \end{pmatrix} = \begin{cases} (-1)^{n(n-1)/2} \det B & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Here O_k stands for the $k \times k$ zero matrix.

Pfaffian

Proposition

For even number n, let

$$S_n = \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & 0 \end{pmatrix}.$$

Then we have

$$\mathrm{Pf}\,(S_n) = 1.$$

That is to say

$$\sum_{\sigma \in \mathscr{F}_n} \mathrm{sgn}\sigma = 1.$$

Minors

Definition

For any finite set S and any nonnegative integer r, let $\binom{S}{r}$ denote the set of all r-element subsets of S. For example, $\binom{[n]}{r}$ stands for the set of all multi-indices $\{i_1, \dots, i_r\}$ such that $1 \le i_1 < \cdots < i_r \le n$. Let n, M and N be positive integers such that $n \le M, N$ and let T be any M by N matrix. For any multi-indices $I = \{i_1, \dots, , i_n\} \in \binom{[M]}{n}$ and $J = \{j_1, \dots, j_n\} \in \binom{[N]}{n}$, let $T^I_J = T^{i_1 \dots i_n}_{j_1 \dots j_n}$ be the sub-matrix of T obtained by picking up the rows indexed by I and the columns indexed by J, i.e.,

$$T^I_J = \begin{pmatrix} t_{i_1 j_1} & \dots & t_{i_1 j_n} \\ \vdots & \ddots & \vdots \\ t_{i_n j_1} & \dots & t_{i_n j_n} \end{pmatrix}.$$

In the case of $n = M$ and $I = [M]$, we omit I from the above expression and write T_J for T^I_J, when there is no possibility of confusion. Similarly we may write T^I for T^I_J if n=N and $J = [N]$.

Summation formulas of Pfaffians

Theorem

Let m and $N = 2N'$ be even integers such that $m \leq N$. Let $T = (t_{ik})_{1 \leq i \leq m, 1 \leq k \leq N}$ be an m by N rectangular matrix. Let $A = (a_{ij})_{1 \leq i,j \leq N}$ be a non-singular skew-symmetric matrix of size N. Then

$$\sum_{I \in \binom{[N]}{m}} \mathrm{Pf}(A_I^I) \det(T_I) = \mathrm{Pf}(Q)$$

Here $Q = (Q_{ij}) = TA\,{}^tT$, and its entries are given by

$$Q_{ij} = \sum_{1 \leq k < l \leq N} a_{kl} \det(T_{kl}^{ij}), \qquad (1 \leq i,j \leq m). \tag{1}$$

Symmetric plane partitions in $B(r, r, s)$

Assume a symmetric plane partition π is in $B(r, r, s)$. For example, the following π is in $B(5, 5, 4)$:

4	4	4	3	2
4	3	3	2	2
4	3	2	2	2
3	2	2	1	1
2	2	2	1	0

4	4	4	3	2
	3	3	2	2
		2	2	2
			1	1
				0

Add r to each entry in the first row, add $r - 1$ to each entry in the second row, and so on, then we obtain the following column-strict plane partition π':

8	8	8	7	6
	6	6	5	5
		4	4	4
			2	2
				0

Symmetric plane partitions in $B(r,r,s)$

We consider the generating function

$$\mathrm{GF}(r,r,s) = \sum_{\pi \subseteq B(r,r,s)} q^{\text{\# of orbits in } \pi}.$$

Assume r is even. Then we have

$$\mathrm{GF}(r,r,s) = q^{-\frac{(r-1)r(r+1)}{3}} \sum_{0 \le k_1 < k_2 < \cdots < k_r \le r+s-1} \det\left(q^{k_j} \begin{bmatrix} i+k_j-1 \\ i-1 \end{bmatrix}_q \right)_{1 \le i,j \le r}.$$

Using the summation formula, we obtain

$$\mathrm{GF}(r,r,s) = q^{-\frac{(r-1)r(r+1)}{3}} \mathrm{Pf}(Q_{ij})_{1 \le i,j \le r},$$

where

$$Q_{ij} = \sum_{0 \le k < l \le r+s-1} \begin{vmatrix} q^k \begin{bmatrix} i+k-1 \\ i-1 \end{bmatrix}_q & q^l \begin{bmatrix} i+l-1 \\ i-1 \end{bmatrix}_q \\ q^k \begin{bmatrix} j+k-1 \\ j-1 \end{bmatrix}_q & q^l \begin{bmatrix} j+l-1 \\ j-1 \end{bmatrix}_q \end{vmatrix}$$

Problem

Show that

$$\mathrm{GF}(r, r, s) = \prod_{x} \frac{1 - q^{\mathrm{ht}(p)+1}}{1 - q^{\mathrm{ht}(p)}}$$

where the product runs over a representative of each orbit in $B(r, r, s)$.

Definition

A plane partition contained in $B(2n,2n,2n)$ is said to be *totally symmetric self-complementary plane parition of size n* if it is totally symmetric and $(2n,2n,2n)$-self-complementary.

We denote the set of all self-complementary totally symmetric plane partitions of size n by $\mathscr{S}_n$.

Example

$\mathscr{S}_1$ consists of the single partition

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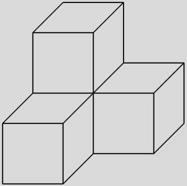
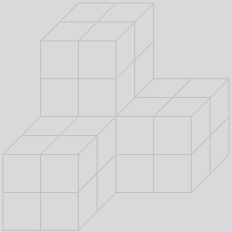
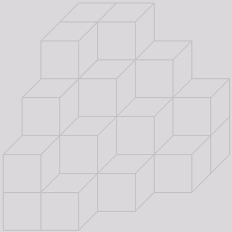


Example

$\mathscr{S}_2$ consists of the following two partitions:





TSSCPPs of size 2

Example

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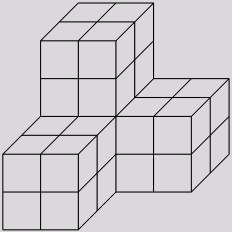
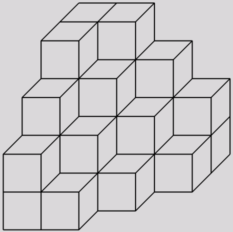
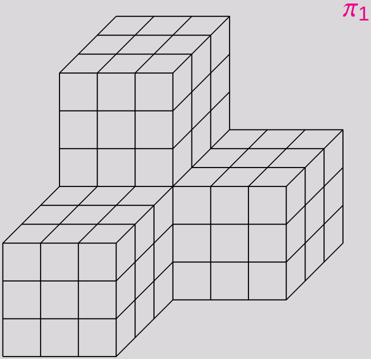
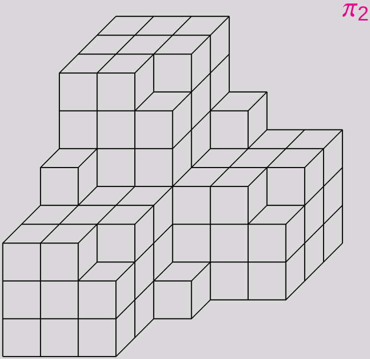




Example

$\mathscr{S}_3$ consists of the following seven partitions:





TSSCPPs of size 3

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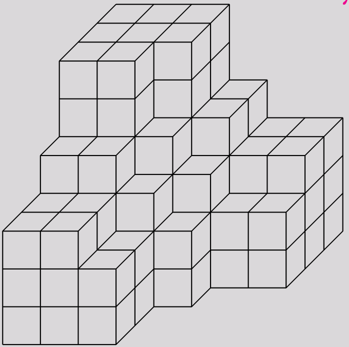
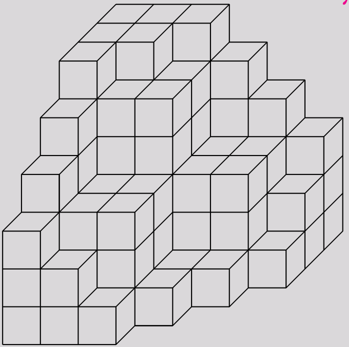
$\mathscr{S}_3$ consists of the following seven partitions:

π_3



π_4



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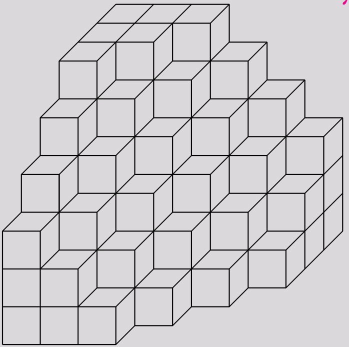
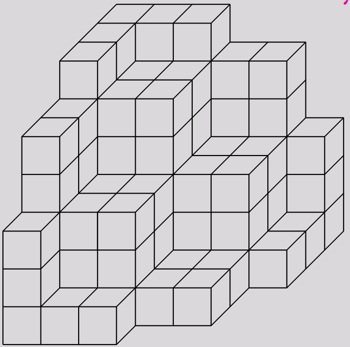


π_5



π_6

TSSCPPs of size 3

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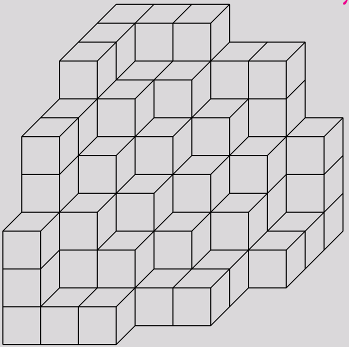

π_7

Definition

A plane partition in $B(2n, 2n, 2n)$ is defined to be *tc-symmetric of size n* if it is cyclically symmetric and it is equal to its transpose-complement.

We denote the set of all tc-symmetric plane partitions of size n by $\mathscr{C}_n$.

Example

$\mathscr{C}_1$ consists of the single partition

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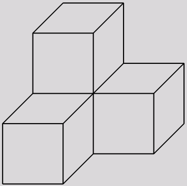

Tc-symmetric PPs of size 2

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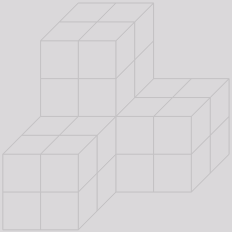

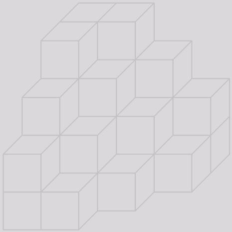

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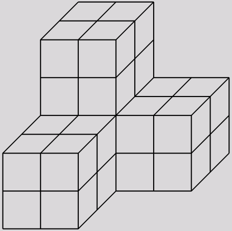
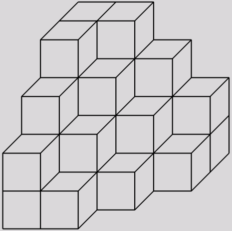
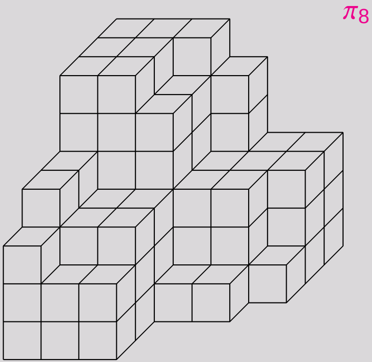
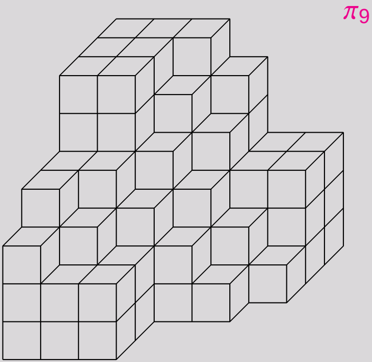




Example

$\mathscr{C}_3$ consists of the following eleven plane partitions:





Numbers

n	1	2	3	4	5	6	$\cdots$
TSSCPP	1	2	7	42	429	7436	$\cdots$
tc-symmetric PP	1	2	11	170	7429	920460	$\cdots$

Definition

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

$$TC_n = \prod_{i=0}^{n-1} \frac{(3i+1)(6i)!(2i)!}{(4i)!(4i+1)!}$$

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1\leq i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n-j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.
A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

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Restricted column-strict plane partitions

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Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*. A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

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$\mathscr{P}_1$ consists of the single element $\emptyset$.

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Example

$\mathscr{P}_2$ consists of the following 2 elements:

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Example

$\mathscr{P}_3$ consists of the followng 7 elements:

$$\emptyset \qquad \boxed{1} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \boxed{2} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

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Pairs of Restricted column-strict plane partitions

Definition

Let $\mathscr{Q}_n$ denote the set of all pairs of plane partitions in $\mathscr{P}_n$ of the same shape.

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$\mathscr{P}_1$ consists of the single pair $(\emptyset, \emptyset)$.

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$\mathscr{P}_2$ consists of the following 2 pairs:

$$(\emptyset, \emptyset) \qquad \left(\boxed{1}, \boxed{1}\right)$$

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$\mathscr{P}_3$ consists of the followng 11 pairs

$$(\emptyset,\emptyset) \quad \left(\begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right) \quad \left(\begin{array}{|c|}\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right) \quad \left(\begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline\end{array}\right) \quad \left(\begin{array}{|c|}\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline\end{array}\right)$$

$$\left(\begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}\right) \quad \left(\begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array}\right) \quad \left(\begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}\right) \quad \left(\begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 2 & 1\\\hline\end{array}\right)$$

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$$(\emptyset,\emptyset) \quad \left(\boxed{1},\boxed{1}\right) \quad \left(\boxed{2},\boxed{1}\right) \quad \left(\boxed{1},\boxed{2}\right) \quad \left(\boxed{2},\boxed{2}\right)$$

$$\left(\boxed{1}\boxed{1},\boxed{1}\boxed{1}\right) \quad \left(\boxed{1}\boxed{1},\boxed{2}\boxed{1}\right) \quad \left(\boxed{2}\boxed{1},\boxed{1}\boxed{1}\right) \quad \left(\boxed{2}\boxed{1},\boxed{2}\boxed{1}\right)$$

$$\left(\begin{array}{l}\boxed{2}\\\boxed{1}\end{array},\begin{array}{l}\boxed{2}\\\boxed{1}\end{array}\right) \quad \left(\begin{array}{l}\boxed{2}\boxed{1}\\\boxed{1}\end{array},\begin{array}{l}\boxed{2}\boxed{1}\\\boxed{1}\end{array}\right)$$

Bijections

Theorem

Let n be a positive integer.
Then we can construct a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Theorem

Let n be a positive integer.
Then we can construct a bijection from $\mathscr{C}_n$ to $\mathscr{Q}_n$.

Example ($n = 3$)

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Example ($n = 3$)

There is 1 RCSPP of shape $\emptyset$.

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There are 2 RCSPPs of shape $\square$:

$\boxed{1}$ $\quad$ $\boxed{\color{red}2}$

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1	1

2	1

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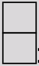
There is 1 RCSPP of shape :

2
1

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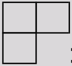
There is 1 RCSPP of shape  :

2	1
1	

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This implies

$$1 + 2 + 2 + 1 + 1 = 7$$
$$1^2 + 2^2 + 2^2 + 1^2 + 1^2 = 11$$

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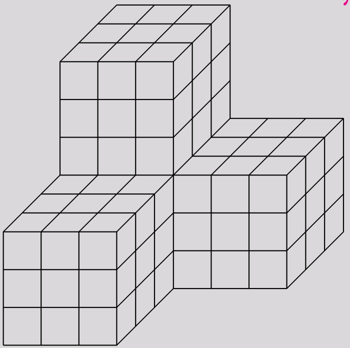
Example

π_1 $\qquad$ $n = 3$



$\longrightarrow \quad \emptyset$

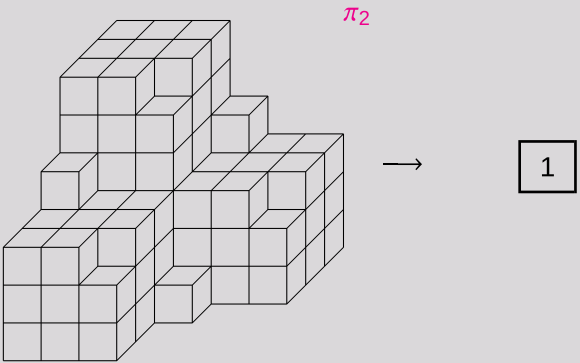
Bijection

Theorem

Let n be a positive integer.
Then we can consruct a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

$n = 3$



Bijection

Theorem

Let n be a positive integer.
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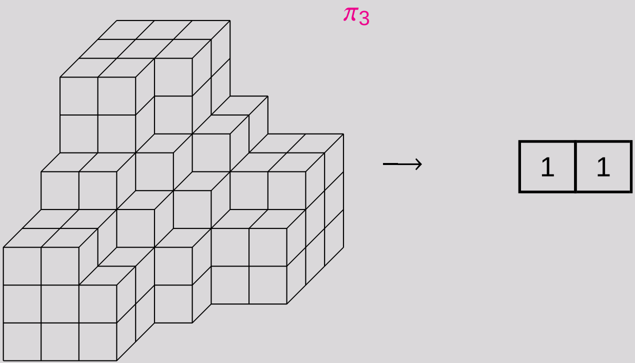
Example

$n = 3$



Bijection

Theorem

Let n be a positive integer.
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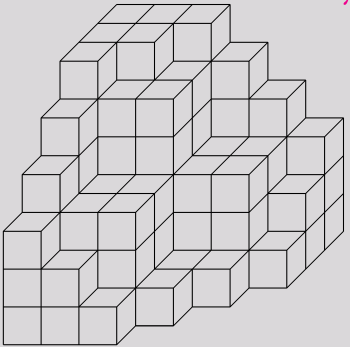
Example

π_4

$n = 3$



$\longrightarrow$

$\boxed{2}$

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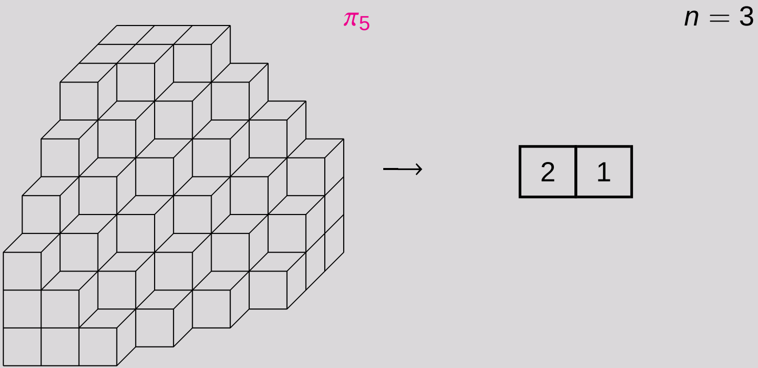
Example



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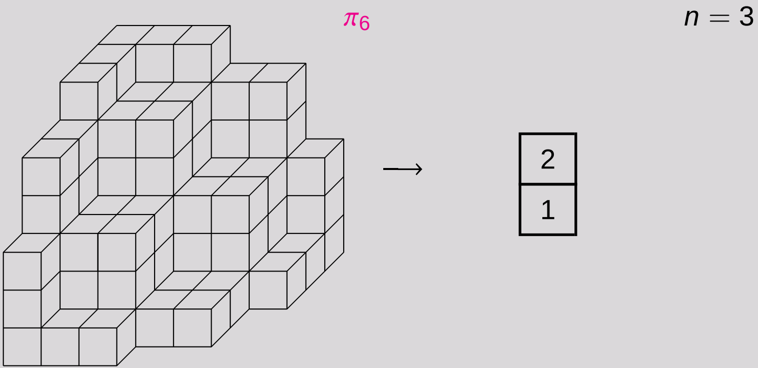

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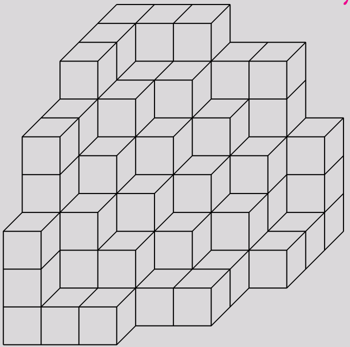
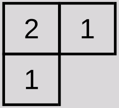
Example

π_7 $\quad$ $n = 3$



$\longrightarrow$



More General Definition

Definition

Let $\mathscr{P}_{n,m}$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

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(C3) c has at most n columns.

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Example

$\mathscr{P}_{0,4}$ consists of the followng 1 element:

$$\emptyset$$

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Example

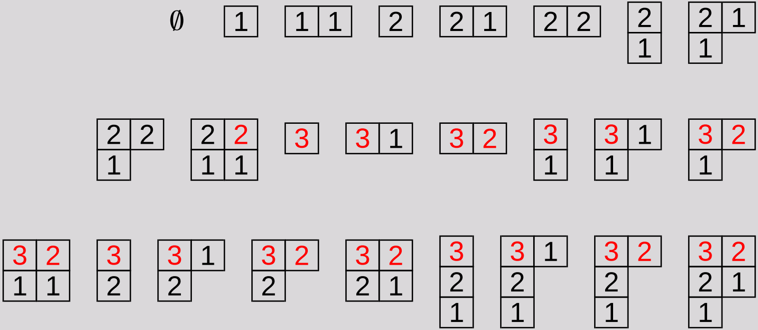
$\mathscr{P}_{1,3}$ consists of the followng 8 elements:

$$\emptyset \quad \boxed{1} \quad \boxed{2} \quad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \quad \boxed{3} \quad \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$$

More General Definition

Example

$\mathscr{P}_{2,2}$ consists of the followng 25 elements:



$\mathscr{P}_{3,1} = \mathscr{P}_{4,0}$ consists of 42 elements.

Example of lattice paths

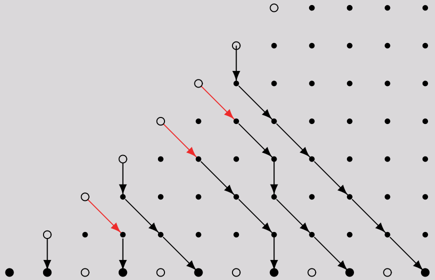
Example

$n = 7$, $c \in \mathscr{P}_7$: RCSPP

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Example

Lattice paths



From RCSPPs to lattice paths

Theorem

Let $V=\{(x,y)\in\mathbb{N}^2 : 0\le y\le x\}$ be the vertex set, and direct an edge from u to v whenever $v-u=(1,-1)$ or $(0,-1)$. Let $u_j=(n-j,n-j)$ and $v_j=(\lambda_j+n-j,0)$ for $j=1,\dots,n$, and let $\mathbf{u}=(u_1,\dots,u_n)$ and $\mathbf{v}=(v_1,\dots,v_n)$. We claim that the $c\in\mathscr{P}_n$ of shape λ' can be identified with n-tuples of nonintersecting D-paths in $\mathscr{P}(\mathbf{u},\mathbf{v})$.

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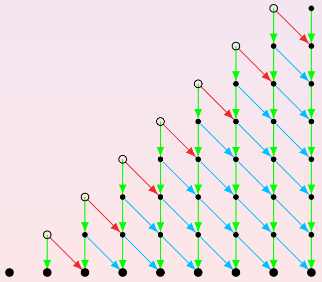

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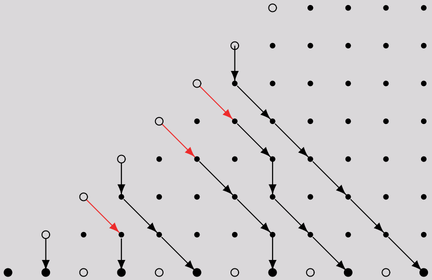
Lattice paths



Generating function

Theorem

Let n be a positive integer. Let λ be a partition such that $\ell(\lambda) \leq n$. Then the generating function of all plane partitions $c \in \mathscr{P}_n$ of shape λ' with the weight $\boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^{c}$ is given by

$$\sum_{\substack{c \in \mathscr{P}_n \\ \mathrm{sh}\, c = \lambda'}} \boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^{c} = \det\left(e^{(n-i)}_{\lambda_j - j + i}(t_1 x_1, \dots, t_{n-i-1} x_{n-i-1}, T_{n-i} x_{n-i}) \right)_{1 \leq i,j \leq n},$$

where $T_i = \prod_{k=i}^{n} t_k$.

$\emptyset$	$\boxed{1}$	$\boxed{1}\boxed{1}$	$\boxed{2}$	$\boxed{2}\boxed{1}$	$\begin{array}{\|c\|}\hline 2 \\ \hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$
1	$t_1 x_1$	$t_1^2 t_2 t_3 x_1^2$	$t_2 t_3 x_1 x_2$	$t_1 t_2 t_3 x_1 x_2$	$t_1 t_2 t_3 x_1 x_2$	$t_1^2 t_2^2 t_3^2 x_1^2 x_2$

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where $T_i = \prod_{k=i}^{n} t_k$.

$\emptyset$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&1\\\hline 1\\\cline{1-1}\end{array}$
1	t_1x_1	$t_1^2t_2t_3x_1^2$	$t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1^2t_2^2t_3^2x_1^2x_2$

Generating function

Theorem

Let n be a positive integer. Let λ be a partition such that $\ell(\lambda) \leq n$. Then the generating function of all plane partitions $c \in \mathscr{P}_n$ of shape λ' with the weight $\boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c$ is given by

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$\emptyset$	1	1 1	2	2 1	2 / 1	2 1 / 1
1	t_1x_1	$t_1^2t_2t_3x_1^2$	$t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1^2t_2^2t_3^2x_1^2x_2$

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A determinant expression

Using Binet-Caucy formula, we obatin the following theorem:

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The number of tc-symmetric plane partitions of size n equals

$$\det\left[\binom{i+j}{2j-i}\right]_{0\le i,j\le n-1}.$$

Theorem

More generally, let $\mathscr{Q}_{n,x,y}$ denote the set of pairs (c_1, c_2) such that $c_1 \in \mathscr{P}_{n,x}$, $c_2 \in \mathscr{P}_{n,y}$, and c_1 and c_2 have the same shape. Then we have

$$\sum_{(c_1,c_2)\in\mathscr{Q}_{n,x,y}} \tau^{|\mathrm{sh}\, c_1|+|\mathrm{sh}\, c_2|} = \det\left[\sum_k \binom{i+x}{k-i}\binom{j+y}{k-j}\tau^{2k-i-j}\right]_{0\le i,j\le n-1}.$$

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Theorem (Ohta-Wenzel formula)

Let A be an anti-symmetric matrix, and let $i_1, \cdots, i_p, j_1, \cdots, j_q$ and $k_1, \cdots k_r$ be row/column indices of A. Then we have

$$\sum_{\alpha=1}^{p} (-1)^{\alpha} \mathrm{Pf} A(i_1, \cdots, \widehat{i_\alpha}, \cdots, i_p, k_1, \cdots, k_r) \mathrm{Pf} A(i_\alpha, j_1, \cdots, j_q, k_1, \cdots, k_r)$$
$$= \sum_{\beta=1}^{q} (-1)^{\beta} \mathrm{Pf} A(i_1, \cdots, i_p, k_1, \cdots, k_r, j_\beta) \mathrm{Pf} A(j_1, \cdots, \widehat{j_\beta}, \cdots, j_q, k_1, \cdots, k_r).$$

Here $\widehat{i}$ means removing i from the original sequence.

Plücker relations for determinants

Theorem

If we apply the Ohta-Wenzel formula to the antisymmetric matrix $A = \begin{pmatrix} O & B \\ -{}^tB & O \end{pmatrix}$, then we obtain the following well-known Plücker relations

$$\sum_{\alpha=1}^{r+1} (-1)^{\alpha} \det B([r]; i_1, \cdots, \widehat{i_\alpha}, \cdots, i_{r+1}) \det B([r]; i_\alpha, j_1, \cdots, j_{r-1}) = 0.$$

As a special case we obtain the following Desnanot-Jacobi formula.

Desnanot–Jacobi formula

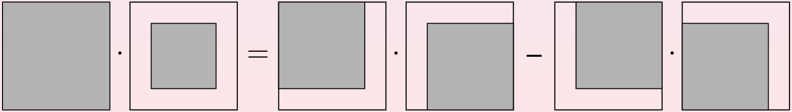
Theorem (Desnanot–Jacobi formula)

Given a matrix M, let

$$M^i_j = \text{the submatrix of } M \text{ obtained by removing row } i \text{ and column } j,$$
$$M^{i,k}_{j,l} = \text{the submatrix of } M \text{ obtained by removing row } i, \text{ row } k, \text{ column } j, \text{ and column } l.$$

Then the Desnanot–Jacobi formula is

$$\det M \cdot \det M^{1,n}_{1,n} = \det M^n_n \cdot \det M^1_1 - \det M^n_1 \cdot \det M^1_n.$$



The matrix

$$\begin{pmatrix} \sum_k \binom{x}{k}\binom{y}{k}\tau^{2k} & \sum_k \binom{x}{k}\binom{y+1}{k-1}\tau^{2k-1} & \cdots \\ \sum_k \binom{x+1}{k-1}\binom{y}{k}\tau^{2k-1} & \sum_k \binom{x+1}{k-1}\binom{y+1}{k-1}\tau^{2k-2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Hirota-Miwa equation

Definition

Let

$$f_{n,x,y} = \det\left[\sum_k \binom{i+x}{k-i-x}\binom{j+y}{k-j-y}\tau^{2k-i-j-x-y}\right]_{0\le i,j\le n-1}.$$

Theorem (Hirota-Miwa equation)

Then $f_{n,x,y}$ satisfies the following equation:

$$f_{n,x,y}f_{n-2,x+1,y+1} = f_{n-1,x,y}f_{n-1,x+1,y+1} - f_{n-1,x+1,y}f_{n,x,y+1},$$

$$f_{0,x,y} = 1, \qquad f_{1,x,y} = \sum_k \binom{x}{k-x}\binom{y}{k-y}\tau^{2k-x-y}.$$

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