平面分割の数え上げ問題と行列式・パフィアン

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Introduction

Abstract

平面分割 (plane partitions) の数え上げ問題は MacMahon が研究を始めて以来、古典的な離散数学の問題として研究されてきたが、対称関数・群の表現論・数理物理などの分野にも現れる組合せ論的側面の研究対象でもある。この話の中では、MacMahon に始まる平面分割の母関数の古典論から始めていろいろな対称性を考慮した平面分割の母関数を、対称関数の応用して得る方法について述べ、その表現論や組合せ論との関係を振り返る。さらに、それらの応用として Mills-Robbins-Rumsey によって提出された totally symmetric self-complementary plane partitions や cyclically symmetric transpose-complementary plane partitions など交代符号行列 (alternating sign matrix) との関連を予想される平面分割の数え上げ問題を扱うことを目標にする。それらの母関数として、行列式・パフィアンによる表示や constant term による表示が得られるが、それらの行列式・パフィアンの計算は Plucker 関係式や discrete Hirota equation などの可積分系との深い関連が予想される。また、最近では affine Hecke algebra などの代数的側面との関係も数理物理学者達によって研究されている。

Plan of My Talk

1. Symmetric functions and
2. Plane partitions and symmetries
3. Totally symmetric self-complementary plane partitions

Symmetric Functions

- I. G. Macdonald, *“Symmetric Functions and Hall Polynomials, 2nd Edition”*, Oxford University Press, (1995)
- A. Lascoux, “Symmetric Functions and Combinatorial Operators on Polynomials”, AMS CBMS **99**.
- R. Stanley, “Theory and Application of Plane Partitions: Part 1,2”, Stud. Appl. Math. **1**, 167 – 188, 259 – 279.

Partition

Partition

A *partition* λ of n is, by definition, a nonincreasing sequence of positive integers

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$$

satisfying $\sum_i \lambda_i = n$. We say λ has $r = \ell(\lambda)$ parts. Similarly a partition of n into distinct parts may be regarded as strictly decreasing sequence of positive integers

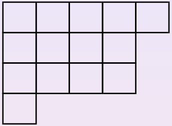
$$\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0.$$

Such a partition is called a *strict partition* of n. We denote partitions in two ways:

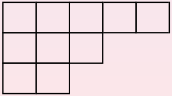
1. $\lambda = (\lambda_1, \lambda_2, \dots)$ signifies that the parts of λ are $\lambda_1 \geq \lambda_2 \geq \cdots$,
2. $\lambda = \langle 1^{r_1} 2^{r_2} \dots \rangle$ signifies that exactly r_i parts of λ are are equal to i.

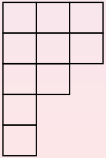
Conjugate

The *diagram* of a partition λ can be defined as the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \le j \le \lambda_i$. For example, $\lambda = (5441)$ is a partition of 14 with 4 parts whose diagram is as follows:



Let $\mathscr{P}_n$ denote the set of partitions of n. If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition of n, let λ' denote the partition whose ith part λ'_i is defined as $\lambda'_i = \#\{j : \lambda_j \ge i\}$. We call λ' the *conjugate partition* of λ. For example, when $\lambda = \langle 235 \rangle$, then $\lambda' = \langle 1^2 2 3^2 \rangle$.





Generating functions

Theorem

The generating function of all partitions is given by

$$\sum_{n\geq 0} \#(\mathscr{P}_n)\, q^n = \prod_{k\geq 1} \frac{1}{1-q^k}.$$

Let $\mathscr{O}_n$ denote the set of $\lambda \in \mathscr{P}_n$ whose parts are all odd, and let $\mathscr{D}_n$ denote the set of strict partitions of n. Then we have

$$\sum_{n\geq 0} \#(\mathscr{O}_n)\, q^n = \sum_{n\geq 0} \#(\mathscr{D}_n)\, q^n = \prod_{k\geq 1} (1+q^k).$$

Dominance order

Given a vector $v \in \mathbb{N}^n$, its *cumulative sum* $\overline{v}$ is the vector

$$\overline{v} = (\overline{v}_1, \dots, \overline{v}_n) = (v_1, v_1 + v_2, \dots, v_1 + v_2 + \cdots + v_n).$$

Definition

Given two partitions λ and μ of the same integer n, $\lambda \leq \mu$ in *dominance order* iff

$$\overline{\lambda_1} \leq \overline{\mu_1},\ \overline{\lambda_2} \leq \overline{\mu_3},\ \dots, \overline{\lambda_n} \leq \overline{\mu_n}.$$

Lemma

An element $u \in \mathbb{N}^n$ is the cumulative sum of a partition iff

$$u_i \geq \frac{u_{i-1} + u_{i+1}}{2}, \qquad 1 < i < n.$$

Lattice Structure

Definition

Let λ and μ be partitions of n. The infimum $\lambda \wedge \mu$ of λ and μ is the partition ν such that $\overline{\nu} = \inf(\overline{\lambda}, \overline{\mu})$. One defines the supremum $\lambda \vee \mu$ by

$$\lambda \vee \mu = (\lambda' \wedge \mu')' .$$

Example

If we take $\lambda = (5111)$ and $\mu = (422)$, then $\overline{\lambda} = (5678)$, $\overline{\mu} = (4688)$, $\inf\left(\overline{\lambda}, \overline{\mu}\right) = (4678) = \overline{\nu}$, with $\nu = (4211) = \lambda \wedge \mu$.
Meanwile, one has $\lambda' = (41111)$, $\mu' = (3311)$, $\overline{\lambda'} = (45678)$, $\overline{\mu'} = (36788)$ and $\inf\left(\overline{\lambda'}, \overline{\mu'}\right) = (35678)$ which implies $\lambda' \wedge \mu' = (32111)$ and $\lambda \vee \mu = (521)$.

Lattice

If the binary operations $\wedge$ and $\vee$ are defined in a set P and satisfy the following axioms, then we call $(P, \wedge, \vee)$ a *lattice*.

1. $x \wedge x = x$, $x \vee x = x$. (Idempotent law)
2. $x \wedge y = y \wedge x$, $x \vee x = y \vee x$. (Commutative law)
3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \vee (y \vee z) = (x \vee y) \vee z$. (Associative law)
4. $x \wedge (x \vee y) = x \vee (x \wedge y) = x$. (Absorption law)

P is said to be *modular* if it satisfies

$$\text{If } x \leq z, \text{ then } x \vee (y \wedge z) = (x \vee y) \wedge z.$$

P is said to be *distributive* if it satisfies

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Dominance order

Proposition

The operations $\wedge$ $\vee$ defines a lattice structure on $\mathscr{P}_n$ such that

$$\lambda \geq \mu \iff \lambda \wedge \mu = \mu \iff \lambda \vee \mu = \lambda.$$

$(\mathscr{P}_n, \leq)$ is a poset (partially ordered set) which is not graded (ranked).

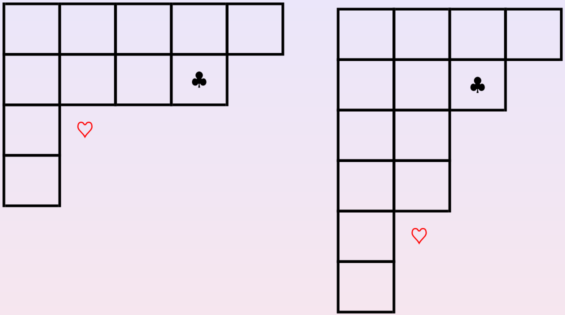
We say λ covers μ iff $\lambda > \mu$ and there is no ν such that $\lambda > \nu > \mu$.

Lemma

Let $\lambda, \mu \in \mathscr{P}_n$. If λ covers *mu* then either

1. $\exists p, k \in \mathbb{N} : \lambda = \nu, p^{k+2}, \varphi \;\&\; \mu = \nu, p+1, p^k, p-1, \varphi$
2. $\exists p, q \in \mathbb{N} : \lambda = \nu, p+1, q-1, \varphi \;\&\; \mu = \nu, p, q, \varphi$

Dominance order



Corollay

If $\lambda, \mu \in \mathscr{P}_n$, then

$$\lambda \geq \mu \Longleftrightarrow \lambda' \leq \mu'.$$

Partitions

Lemma

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition contained in m^n, and $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ its conjugate. Then $\{\lambda_1 + n - 1, \ldots, \lambda_n\}$ and $\{n - \lambda'_1, \ldots, n + m - 1 - \lambda'_m\}$ are complementary sets in $\{0, \ldots, m + n - 1\}$.

Example

For example, take $m = 5$, $n = 4$, $\lambda = (4, 2, 0, 0)$, then one obtains $\lambda' = (2, 2, 1, 1, 0)$ which implies $\{\lambda_i + n - i\} = \{0, 1, 4, 7\}$ and $\{n - 1 + i - \lambda'_i\} = \{2, 3, 5, 6, 8\}$.

Hook Length and Contents

Definition

For $x = (i,j) \in \lambda = (\lambda_1, \dots, \lambda_r) \in \mathscr{P}_n$, one defines the *hook length* $h(i,j) = \lambda_i - j + \lambda'_j - i + 1$ and the *content* $c(i,j) = j - i$.

10	9	7	4	3	2
9	8	6	3	2	1
5	4	2			
4	3	1			
2	1				

0	1	2	3	4	5
−1	0	1	2	3	4
−2	−1	0			
−3	−2	−1			
−4	−3				

Symmetric Functions

A symmetric function of an alphabet $\mathbb{A}$ is a function of the letters which is invariant under permutations of letters of $\mathbb{A}$. We use the notation $\mathbb{A} + \mathbb{B} = \mathbb{A} \cup \mathbb{B}$.

Definition

One defines the *elementary symmetric functions* $\Lambda^i(\mathbb{A})$, the *complete symmetric functions* $S^i(\mathbb{A})$ and the *power sums* $\Psi_i(\mathbb{A})$ by

$$\lambda_z(\mathbb{A}) = \sum_{i \geq 0} z^i \Lambda^i(\mathbb{A}) = \prod_{a \in \mathbb{A}} (1 + za),$$

$$\sigma_z(\mathbb{A}) = \sum_{i \geq 0} z^i S^i(\mathbb{A}) = \prod_{a \in \mathbb{A}} \frac{1}{1 - za},$$

$$\boldsymbol{\Psi}_z(\mathbb{A}) = \sum_{i \geq 1} \frac{z^i}{i} \Psi^i(\mathbb{A}) = \sum_{i \geq 1} \sum_{a \in \mathbb{A}} \frac{z^i}{i} a^i.$$

Addition of alphabets

We have

$$\sigma_z(\mathbb{A}) = \exp(\boldsymbol{\Psi}_z(\mathbb{A})), \qquad \boldsymbol{\Psi}_z(\mathbb{A}) = \log(\sigma_z(\mathbb{A})).$$

Definition

One extends by

$$\lambda_z(\mathbb{A}-\mathbb{B}) = \sum_{i\geq 0} z^i \Lambda^i(\mathbb{A}-\mathbb{B}) = \frac{\prod_{a\in\mathbb{A}}(1+za)}{\prod_{b\in\mathbb{B}}(1+zb)}$$

$$\sigma_z(\mathbb{A}-\mathbb{B}) = \sum_{i\geq 0} z^i S^i(\mathbb{A}-\mathbb{B}) = \frac{\prod_{b\in\mathbb{B}}(1-zb)}{\prod_{a\in\mathbb{A}}(1-za)}.$$

Thus we have

$$S^k(-\mathbb{B}) = (-1)^k \Lambda^k(\mathbb{B}), \qquad \Lambda^k(-\mathbb{B}) = (-1)^k S^k(\mathbb{B}).$$

The Ring of Symmetric Functions

Given a finite alphabet $\mathbb{A}$, let $\mathscr{S}(\mathbb{A})$ be the ring of symmetric polynomials overe the rational numbers.

Definition

As vector space, it has bases

$$\begin{cases} \Lambda^{\lambda}(\mathbb{A}) = \Lambda^{\lambda_1}(\mathbb{A})\Lambda^{\lambda_2}(\mathbb{A})\cdots \\ S^{\lambda}(\mathbb{A}) = S^{\lambda_1}(\mathbb{A})S^{\lambda_2}(\mathbb{A})\cdots \\ \Psi^{\lambda}(\mathbb{A}) = \Psi^{\lambda_1}(\mathbb{A})\Psi^{\lambda_2}(\mathbb{A})\cdots \end{cases},$$

where k runs over all integers in $\mathbb{N}$, and λ runs over all partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \leq \#(\mathbb{A})$.

Theorem (Newton)

If $\mathbb{A}$ is an alphabet of cardinality n, then $\mathscr{S}(\mathbb{A})$ is a polynomial ring with generators $\Lambda^1(\mathbb{A}), \dots, \Lambda^n(\mathbb{A})$.

The Ring of Symmetric Functions

Corollary

$S^1(\mathbb{A}),\dots,S^n(\mathbb{A})$ and $\Psi^1(\mathbb{A}),\dots,\Psi^n(\mathbb{A})$ are also algebraic bases of $\mathscr{S}(\mathbb{A})$.

From $\sigma_z(\mathbb{A})\lambda_{-z}(\mathbb{A}) = 1$, one obtains

$$\sum_{r=0}^{n} (-1)^r \Lambda^r(\mathbb{A}) S^{n-r}(\mathbb{A}) = 0$$

for $n \geq 1$.
From $\boldsymbol{\Psi}_z(\mathbb{A}) = \log(\sigma_z(\mathbb{A}))$ and $\sigma_z(\mathbb{A}) = \frac{1}{\lambda_{-z}(\mathbb{A})}$, one obtains

$$n\Lambda^n(\mathbb{A}) = \sum_{r=1}^{n} (-1)^{r-1} \Psi^r(\mathbb{A}) \Lambda^{n-r}(\mathbb{A})$$

for $n \geq 1$.

Matrix Generating Functions

Definition

We define Toelitz matrices $\mathbb{S}(\mathbb{A})$ and $\mathbb{L}(\mathbb{A})$ by

$$\mathbb{S}(\mathbb{A}) = \left[S^{i-j}(\mathbb{A})\right]_{i,j\geq 0}, \qquad \mathbb{L}(\mathbb{A}) = \left[\Lambda^{i-j}(\mathbb{A})\right]_{i,j\geq 0}.$$

Addition or subtraction of alphabets correspond to product of matrix:

$$\mathbb{S}(\mathbb{A} \pm \mathbb{B}) = \mathbb{S}(\mathbb{A})\,\mathbb{S}(\mathbb{B})^{\pm 1}, \quad \mathbb{L}(\mathbb{A} \pm \mathbb{B}) = \mathbb{L}(\mathbb{A})\,\mathbb{L}(\mathbb{B})^{\pm 1}.$$

Definition

For a partition λ such that $\ell(\lambda) \leq n$, let $J_n(\lambda) = \{\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_n\}$. Given partitions λ and μ, one defines the *skew Schur functions* $S_{\lambda/\mu}(\mathbb{A})$ to be the minor of $\mathbb{S}(\mathbb{A})$ taken on rows $J_n(\lambda)$ and columns $J_n(\mu)$. When $\mu = \emptyset$, the minor is called a *Schur function* and one writes $S_\lambda(\mathbb{A})$.

Schur functions

Definition

In other words,

$$S_{\lambda/\mu}(\mathbb{A}) = \det\left[S^{\lambda_i-\mu_j-i+j}(\mathbb{A})\right]_{1\leq i,j\leq n}.$$

It is convenient to also use determinants in elementary symmetric functions

$$\Lambda_{\lambda/\mu}(\mathbb{A}) = \det\left[\Lambda^{\lambda_i-\mu_j-i+j}(\mathbb{A})\right]_{1\leq i,j\leq n}.$$

Definition

We shall denote, by $\mathbb{S}_{\lambda/\mu}(\mathbb{A})$ (resp. $\mathbb{L}_{\lambda/\mu}(\mathbb{A})$), the submatrix of $\mathbb{S}(\mathbb{A})$ (resp. $\mathbb{L}(\mathbb{A})$) taken on rows $J_n(\lambda)$ and columns $J_n(\mu)$.

Binet-Cauchy theorem

Let m, n, r and s be nonnegative integers. Given am $m \times n$ matrix $X = (x_{ij})_{1 \le i \le m, 1 \le j \le n}$, a row index set $I = (i_1, \cdots, i_r)$ and a colu,mn index set $J = (j_1, \cdots, j_s)$, Let $X(I; J)$ denote the $r \times s$ submatrix obtained by choosing the rows $i_1, \cdots, i_r$ and the columns $j_1, \cdots, j_s$ from X:

$$X(I; J) = \left(x_{i_p, j_q}\right)_{1 \le p \le r, 1 \le q \le s}.$$

Let $\binom{[n]}{r}$ denote the set of all r-element subsets of $[n]$.

Theorem (Binet-Cauchy)

Let m and M be positive integers such that $m \le M$. Let B be a square matrix of size M, and R and S be $m \times M$ rectangular matrices. Then we have

$$\sum_{I, J \in \binom{[M]}{m}} \det B(I; J) \det R([m]; I) \det S([m]; J) = \det\left(RB^{t}S\right).$$

Triangular shifted plane partitions

Theorem

The Binet-Cauchy theorem for minors of the product of two matrices implies the following expansion of skew-Schur functions

$$s_{\lambda/\mu}(\mathbb{A}+\mathbb{B}) = \sum_{\nu} s_{\lambda/\nu}(\mathbb{A})\, s_{\nu/\mu}(\mathbb{B}),$$

where the sum runs over all partitions (only those *nu* : $\lambda \supseteq \nu \supseteq \mu$ give nonzero contribution).

Theorem (Jacobi)

Let A be an n by n matrix and $\widetilde{A}$ be its cofactor matrix. Let $r \leq n$ and $I, J \subseteq [n]$, $\#I = \#J = r$. Then

$$\det \widetilde{A}^{I}_{J} = (-1)^{|I|+|J|} \left(\det A\right)^{r-1} \det A^{J^c}_{I^c},$$

where $I^c, J^c \subseteq [n]$ stand for the complements of I, J, respectively, in $\subseteq [n]$. Here we denote $|I| = \sum_{i \in I} i$.

Theorem

Jacobi's theorem for minors of the product of two matrices implies

$$\Lambda_{\lambda/\mu}\left(\mathbb{A}\right) = S_{\lambda'/\mu'}\left(\mathbb{A}\right) = (-1)^{\lambda/\mu} S_{\lambda/\mu}\left(-\mathbb{A}\right),$$

where the sum runs over all partitions (only those $nu : \lambda \supseteq \nu \supseteq \mu$ give nonzero contribution).

Proof

Since $\Lambda^i(\mathbb{A}) = (-1)^i S^i(-\mathbb{A})$ for $i \in \mathbb{Z}$, one immediately obtains

$$\Lambda_{\lambda/\mu}(\mathbb{A}) = (-1)^{\lambda/\mu} S_{\lambda/\mu}(-\mathbb{A}).$$

Meanwhile, since $\mathbb{S}(-\mathbb{A}) = \mathbb{S}(\mathbb{A})^{-1} = \widetilde{\mathbb{S}(\mathbb{A})}$, Jacobi's theorem implies

$$\det \mathbb{S}(-\mathbb{A})(J_n(\lambda); J_n(\mu)) = (-1)^{|\lambda/\mu|} \det \mathbb{S}(\mathbb{A})(J_n(\mu)^c; J_n(\lambda)^c).$$

Use the above lemma.

Multi-Schur functions

Definition

Given n, two sets of alphabets $\{\mathbb{A}_1, \ldots, \mathbb{A}_n\}$, $\{\mathbb{B}_1, \ldots, \mathbb{B}_n\}$, and $\lambda, \mu \in \mathbb{N}^n$, we define the *multi-Schur functions*

$$S_{\lambda/\mu}\left(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n\right) = \det\left(S^{\lambda_i - \mu_j - i + j}\left(\mathbb{A}_i - \mathbb{B}_i\right)\right)_{1 \le i,j \le n}.$$

In the case where the alphabets are repeated, we indicate by a semicolon the corresponding block separation: given $\lambda \in \mathbb{Z}^p$, $\mu \in \mathbb{Z}^q$, then $S_{\lambda;\mu}(\mathbb{A} - \mathbb{B}; \mathbb{C} - \mathbb{D})$ stands for the multi-Schur function with index the concatenation of λ and μ, and $\mathbb{A}_1 = \cdots = \mathbb{A}_p = \mathbb{A}$, $\mathbb{B}_1 = \cdots = \mathbb{B}_p = \mathbb{B}$, $\mathbb{A}_{p+1} = \cdots = \mathbb{A}_{p+q} = \mathbb{C}$, $\mathbb{B}_{p+1} = \cdots = \mathbb{B}_{p+q} = \mathbb{D}$.

$$S_{4;21}\left(\mathbb{A}; \mathbb{B}\right) = \det\begin{pmatrix} S^4\left(\mathbb{A}\right) & S^5\left(\mathbb{A}\right) & S^6\left(\mathbb{A}\right) \\ S^1\left(\mathbb{B}\right) & S^2\left(\mathbb{B}\right) & S^3\left(\mathbb{B}\right) \\ S^{-1}\left(\mathbb{B}\right) & S^0\left(\mathbb{B}\right) & S^1\left(\mathbb{B}\right) \end{pmatrix}$$